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Generalized Geometries in String Compactification Scenarios

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Introduction: Compactifications in String Theories

We are looking for the origin of 4D physics

Physical information

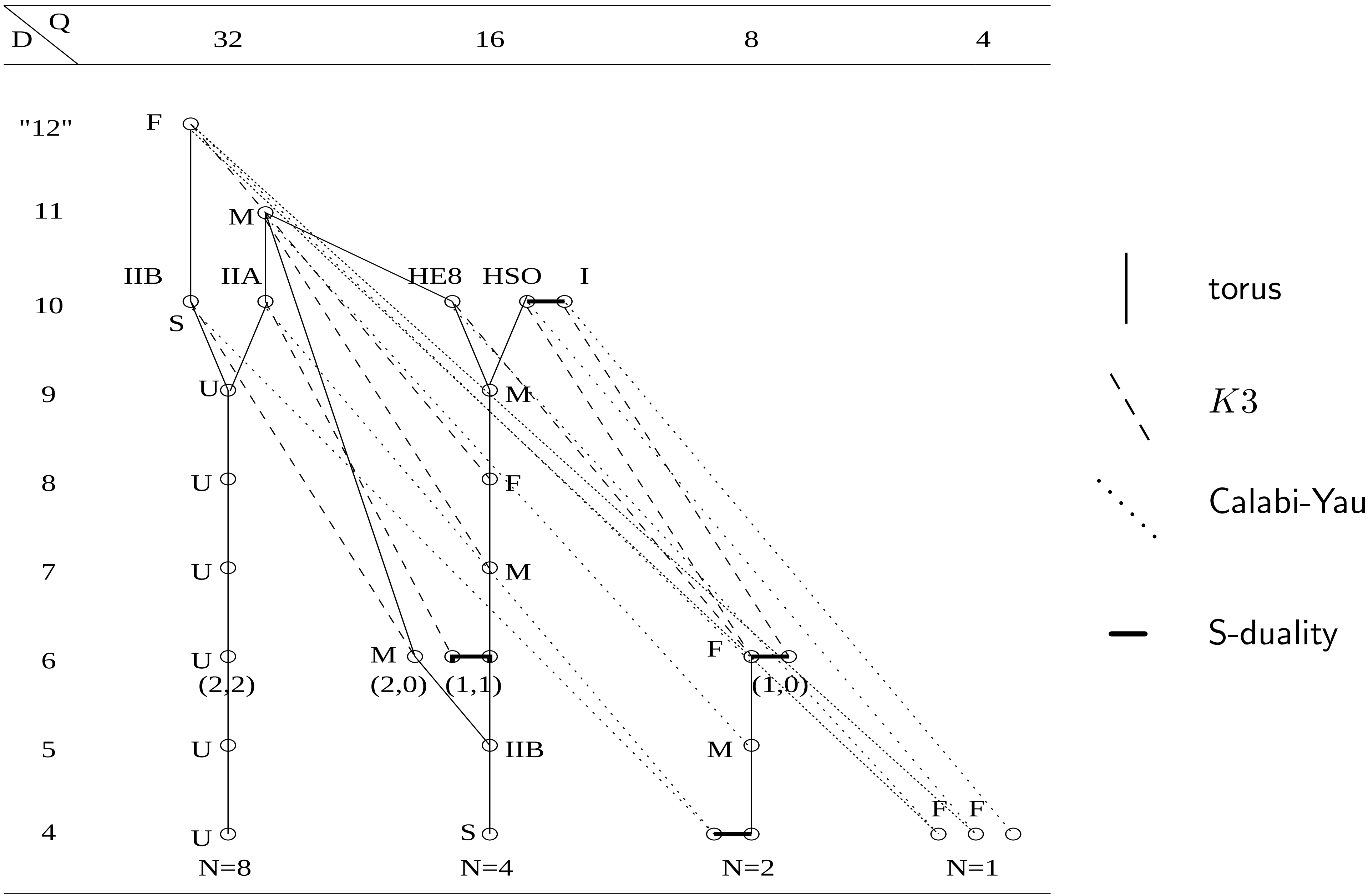
- Particle contents and spectra
- (Broken) symmetries
- Potential, vacuum and cosmological constant

What kind of 4D models come from String Theories?

$$\downarrow$$

What kind of Compactifications?

$$\mathbf{4 \;=\; 10 - 6 \;=\; 11 - 7}$$



B. de Wit and J. Louis, in the Proceedings "NATO Advanced Study Institute on Strings, Branes and Dualities (1997)" hep-th/9801132

- Many Abelian Supergravities (SUGRA) in lower dimensions

  Compactifications on Tori, Calabi-Yaus, etc.

  Minkowski ground state, massless fields

  Global E_7 symmetry (4D $\mathcal{N} = 8$ case)

- Many Gauged SUGRA in lower dimensions

  Compactifications on group manifolds, torsionful manifolds, etc.

  Scalar potential generating masses [Moduli Stabilization]

  Non-trivial cosmological constant

There are various Gauged SUGRA

which cannot be derived from String Theories

compactified on conventional geometric backgrounds

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We want to derive all Gauged SUGRA from String Theories

Compactify String Theories on non-conventional geometries:

There are various Gauged SUGRA

which cannot be derived from String Theories

compactified on conventional geometric backgrounds

$\downarrow$

We want to derive all Gauged SUGRA from String Theories

Compactify String Theories on non-conventional geometries:

Nongeometric String Backgrounds

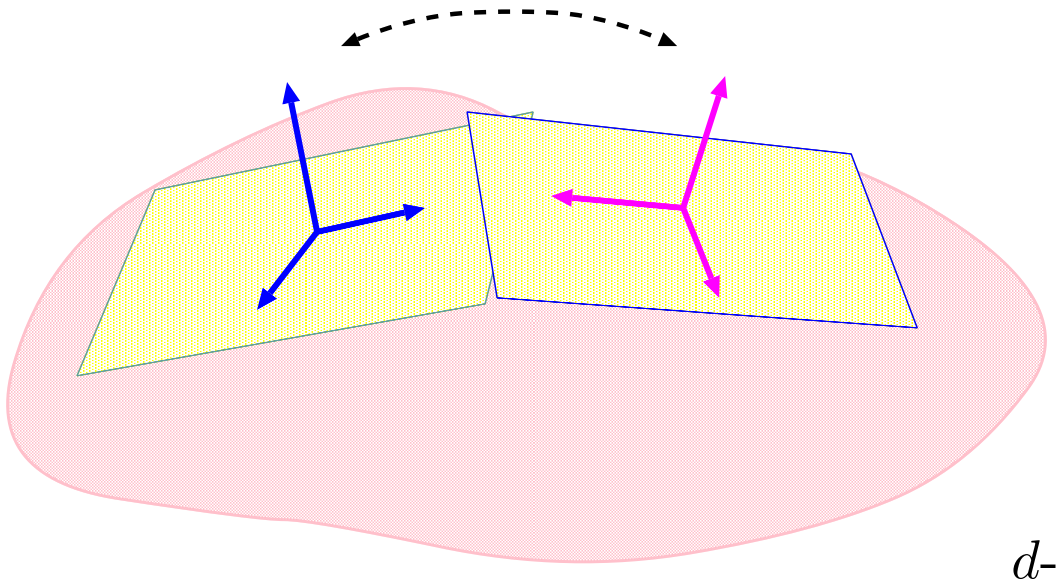
What is a Nongeometric String Background?

Structure group $=$ Diffeo. $(GL(d,\mathbb{R})) + \underbrace{\text{Duality transf. groups}}$

$\uparrow$

coming from String dualities

$GL(d,\mathbb{R})$ + duality transf.



d-dim. internal space $\mathcal{M}_d \simeq$ monodrofold

SUGRA on Nongeometric String Backgrounds

cf.) Lower-dim. Gauged SUGRA compactified by Scherk-Schwarz mechanism

"Kaloper-Myers" algebra:

$$
\begin{aligned}
[Z_a, Z_b] &= f_{ab}{}^c \, Z_c + H_{abc} \, X^c \\
[X^a, X^b] &= Q^{ab}{}_c \, X^c + R^{abc} \, Z_c \\
[X^a, Z_b] &= f^a{}_{bc} \, X^c - Q^{ac}{}_b \, Z_c
\end{aligned}
$$

Various "fluxes" are involved

N. Kaloper, R.C. Myers hep-th/9901045

J. Shelton, W. Taylor, B. Wecht hep-th/0508133, A. Dabholkar, C.M. Hull hep-th/0512005

M. Graña, R. Minasian, M. Petrini, D. Waldram arXiv:0807.4527

String Theories compactified on Nongeometric Backgrounds

$\downarrow$

All(?) Gauged SUGRA

Hitchin's Generalized Geometries to study vacua

Hull's Doubled Formalism to find gauge symmetries

- Calabi-Yau three-folds $\dashrightarrow$ Fluxes are highly restricted

$$\begin{cases} \text{type IIA}: & \text{No fluxes} \\ \text{type IIB}: & F_3 - \tau H \qquad \text{(warped Calabi-Yau)} \\ \text{heterotic}: & \text{No fluxes} \end{cases}$$

- $SU(3)$-structure manifolds $\dashrightarrow$ Some components of fluxes can be interpreted as torsion

Piljin Yi, TK "*Comments on heterotic flux compactifications*" JHEP 0607 (2006) 030, hep-th/0605247

TK "*Index theorems on torsional geometries*" JHEP 0708 (2007) 048, arXiv:0704.2111

- Generalized geometries $\dashrightarrow$ Any types of fluxes can be introduced

"Complete" classification of $\mathcal{N} = 1$ SUSY solutions

Search 4D SUSY vacua in type IIA theory compactified on generalized geometries

- Moduli stabilization

  We find SUSY AdS (or Minkowski) vacua

- Mathematical feature

  We obtain a powerful rule to evaluate vacua:

  Discriminant of the superpotential governs the cosmological constant

- Stringy effects

  We see that α' corrections are included in certain configurations

Contents

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Decomposition of vector bundle on 10D spacetime:

$$TM_{9,1} = T_{3,1} \oplus F$$

$$\begin{cases} T_{3,1} : & \text{a real } SO(3,1) \text{ vector bundle} \\ F : & \text{an } SO(6) \text{ vector bundle which admits a pair of } SU(3) \text{ structures} \end{cases}$$

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Decomposition of Lorentz symmetry:

$$Spin(9,1) \to Spin(3,1) \times Spin(6) = SL(2,\mathbb{C}) \times SU(4)$$

$$\mathbf{16} = (\mathbf{2}, \mathbf{4}) \oplus (\overline{\mathbf{2}}, \overline{\mathbf{4}}) \qquad \mathbf{16} = (\mathbf{2}, \overline{\mathbf{4}}) \oplus (\overline{\mathbf{2}}, \mathbf{4})$$

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Decomposition of supersymmetry parameters (with $a, b \in \mathbb{C}$):

$$\begin{aligned} \epsilon^1_{\mathrm{IIA}} &= \varepsilon_1 \otimes (\overline{a}\, \eta^1_-) + \varepsilon^c_1 \otimes (a\, \eta^1_+) \\ \epsilon^2_{\mathrm{IIA}} &= \varepsilon_2 \otimes (b\, \eta^2_+) + \varepsilon^c_2 \otimes (\overline{b}\, \eta^2_-) \end{aligned}$$

$$\begin{aligned} \epsilon^1_{\mathrm{IIB}} &= \varepsilon_1 \otimes (\overline{a}\, \eta^1_-) + \varepsilon^c_1 \otimes (a\, \eta^1_+) \\ \epsilon^2_{\mathrm{IIB}} &= \varepsilon_2 \otimes (\overline{b}\, \eta^2_-) + \varepsilon^c_2 \otimes (b\, \eta^2_+) \end{aligned}$$

Set $SU(3)$ invariant spinor $\eta^{\mathcal{A}}_+$ s.t. $\nabla^{(T)} \eta^{\mathcal{A}}_+ = 0$ $(\mathcal{A} = 1, 2)$

a pair of $SU(3)$ on F (η^1_+, η^2_+) $\longleftrightarrow$ a single $SU(3)$ on F $(\eta^1_+ = \eta^2_+ = \eta_+)$

- with a single $SU(3)$:

a real two-form	$J_{mn} = \mp 2\mathrm{i}\, \eta_{\pm}^{\dagger}\, \gamma_{mn}\, \eta_{\pm}$
a complex three-form	$\Omega_{mnp} = -2\mathrm{i}\, \eta_{-}^{\dagger}\, \gamma_{mnp}\, \eta_{+}$

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- with a pair of $SU(3)$:

two real vectors	$(v - \mathrm{i}v')^m = \eta_{+}^{1\dagger} \gamma^m\, \eta_{-}^{2}$
(J^A, Ω^A)	$J^1 = j + v \wedge v'\,, \quad \Omega^1 = w \wedge (v + \mathrm{i}v')$ $J^2 = j - v \wedge v'\,, \quad \Omega^2 = w \wedge (v - \mathrm{i}v')$ (j, w): locally $SU(2)$-invariant two-forms

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$$\eta_+^2 \;=\; c_\| \eta_+^1 + c_\perp (v + \mathrm{i}v')^m\, \gamma_m\, \eta_-^1\,, \qquad |c_\||^2 + |c_\perp|^2 = 1$$

If $\eta_+^1 \neq \eta_+^2$ globally: a single $SU(2)$ w/ (j, w, v, v')

If $\eta_+^1 = \eta_+^2$ globally: a single $SU(3)$ w/ (J, Ω)

a pair of $SU(3)$ on $F \;\sim\; SU(3) \times SU(3)$ on $F \oplus F^*$

Information from Killing spinor eqs. with torsion $\nabla^{(T)}\eta_\pm = 0$ ($^\exists$complex Weyl $\eta_\pm$)

- Invariant p-forms on $SU(3)$-structure manifold:

a real two-form $\quad J_{mn} = \mp 2\mathrm{i}\, \eta_\pm^\dagger \, \gamma_{mn} \, \eta_\pm$

a holomorphic three-form $\quad \Omega_{mnp} = -2\mathrm{i}\, \eta_-^\dagger \, \gamma_{mnp} \, \eta_+$

$$\mathrm{d}J = \frac{3}{2}\mathrm{Im}(\overline{\mathcal{W}}_1 \Omega) + \mathcal{W}_4 \wedge J + \mathcal{W}_3 \qquad \mathrm{d}\Omega = \mathcal{W}_1 J \wedge J + \mathcal{W}_2 \wedge J + \overline{\mathcal{W}}_5 \wedge \Omega$$

- Five classes of (intrinsic) torsion are given as

components	interpretations	$SU(3)$-representations
$\mathcal{W}_1$	$J \wedge \mathrm{d}\Omega$ or $\Omega \wedge \mathrm{d}J$	$\mathbf{1} \oplus \mathbf{1}$
$\mathcal{W}_2$	$(\mathrm{d}\Omega)_0^{2,2}$	$\mathbf{8} \oplus \mathbf{8}$
$\mathcal{W}_3$	$(\mathrm{d}J)_0^{2,1} + (\mathrm{d}J)_0^{1,2}$	$\mathbf{6} \oplus \overline{\mathbf{6}}$
$\mathcal{W}_4$	$J \wedge \mathrm{d}J$	$\mathbf{3} \oplus \overline{\mathbf{3}}$
$\mathcal{W}_5$	$(\mathrm{d}\Omega)^{3,1}$	$\mathbf{3} \oplus \overline{\mathbf{3}}$

- Classification of $SU(3)$-structure manifolds:

complex	hermitian	$\mathcal{W}_1 = \mathcal{W}_2 = 0$
	balanced	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_4 = 0$
	special hermitian	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	Kähler	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = 0$
	Calabi-Yau	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	conformally Calabi-Yau	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = 3\mathcal{W}_4 + 2\mathcal{W}_5 = 0$
almost complex	symplectic	$\mathcal{W}_1 = \mathcal{W}_3 = \mathcal{W}_4 = 0$
	nearly Kähler	$\mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	almost Kähler	$\mathcal{W}_1 = \mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	quasi Kähler	$\mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	semi Kähler	$\mathcal{W}_4 = \mathcal{W}_5 = 0$
	half-flat	$\mathrm{Im}\mathcal{W}_1 = \mathrm{Im}\mathcal{W}_2 = \mathcal{W}_4 = \mathcal{W}_5 = 0$

Contents

Introduce a generalized almost complex structure $\mathcal{J}$ on $F \oplus F^*$ s.t.

$$\mathcal{J} : \; F \oplus F^* \longrightarrow F \oplus F^*$$

$$\mathcal{J}^2 = -\mathbf{1}_{2d}$$

$$^{\exists}\, O(d,d) \text{ invariant metric } L, \text{ s.t. } \; \mathcal{J}^T L \mathcal{J} = L$$

Structure group on $F \oplus F^*$			
$^{\exists}L$	$GL(2d)$	$\dashrightarrow$	$O(d,d)$
$\mathcal{J}^2 = -\mathbf{1}_{2d}$	$O(d,d)$	$\dashrightarrow$	$U(d/2,d/2)$
$\mathcal{J}_1, \mathcal{J}_2$	$U_1(d/2,d/2) \cap U_2(d/2,d/2)$	$\dashrightarrow$	$U(d/2) \times U(d/2)$
integrable $\mathcal{J}_{1,2}$	$U(d/2) \times U(d/2)$	$\dashrightarrow$	$SU(d/2) \times SU(d/2)$

- Integrability is discussed by "$(0,1)$" part of the complexified $F \oplus F^*$:

$$\Pi \ \equiv \ \frac{1}{2}(\mathbf{1}_{2d} - \mathrm{i}\mathcal{J})$$

$$\Pi A \ = \ A \quad \text{where } A = v + \zeta \text{ is a section of } F \oplus F^*$$

We call this A i-eigenbundle $L_{\mathcal{J}}$ whose dimension is $\dim L_{\mathcal{J}} = d$.

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Integrability condition of $\mathcal{J}$ is

$$\overline{\Pi}\big[\Pi(v+\zeta), \Pi(w+\eta)\big]_{\text{Courant}} \;=\; 0\,; \qquad v, w \in \text{section of } F\,; \quad \zeta, \eta \in \text{section of } F^*$$

$$\boxed{\big[v+\zeta, w+\eta\big]_{\text{Courant}} \;=\; [v,w]_{\text{Lie}} + \mathcal{L}_v \eta - \mathcal{L}_w \zeta - \frac{1}{2}\mathrm{d}(\iota_v \eta - \iota_w \zeta) \qquad \text{Courant bracket}}$$

- Two examples of generalized almost complex structures:

$$\mathcal{J}_- = \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & -I^T \end{pmatrix} \qquad \text{w/ } I^2 = -\mathbf{1}_d\text{: almost complex structure}$$

$$\mathcal{J}_+ = \begin{pmatrix} \mathbf{0} & -J^{-1} \\ J & \mathbf{0} \end{pmatrix} \qquad \text{w/ } J\text{: almost symplectic form}$$

$$\text{integrable } \mathcal{J}_- \quad \leftrightarrow \quad \text{integrable } I$$

$$\text{integrable } \mathcal{J}_+ \quad \leftrightarrow \quad \text{integrable } J$$

On a usual geometry, $J_{mn} = g_{mp}\, I^p{}_n$ is given by an $SU(3)$ invariant (pure) spinor η_+ as

$$J_{mn} = -2\mathrm{i}\, \eta_+^\dagger \gamma_{mn} \eta_+ \qquad \gamma^i \eta_+ = 0 \qquad \gamma^{\bar{\imath}} \eta_+ \neq 0$$

In a similar analogy, we want to find pure spinor(s) Φ on gengeralized geometry.

On $F \oplus F^*$, we can define Cliff$(6,6)$ algebra and $Spin(6,6)$ spinor Φ:

$$\{\Gamma^m, \Gamma^n\} = 0 \qquad \{\Gamma^m, \widetilde{\Gamma}_n\} = \delta^m_n \qquad \{\widetilde{\Gamma}_m, \widetilde{\Gamma}_n\} = 0$$

Irreducible repr. of $Spin(6,6)$ spinor is a Majorana-Weyl

$\rightarrow$ a generic $Spin(6,6)$ spinor bundle S splits to $S^{\pm}$ (Weyl)

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Weyl spinor bundles $S^{\pm}$ are isomorphic to bundles of forms F^*:

$\Phi_+ \in S^+$	$\sim$	section of $\wedge^{\text{even}} F^*$
$\Phi_- \in S^-$	$\sim$	section of $\wedge^{\text{odd}} F^*$

A form-valued representation of the algebra

$$\Gamma^m = \mathrm{d}x^m \wedge \, , \qquad \widetilde{\Gamma}_n = \iota_{\partial_n}$$

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IF Φ is annihilated by half numbers of the Cliff$(6,6)$ generators:

$\longrightarrow$ Φ is called a pure spinor

cf.) $SU(3)$ invariant spinor η_+ is a pure spinor: $\gamma^i \eta_+ = 0$

An equivalent definition of a pure spinor Φ is given by "Clifford action":

$$(v + \zeta) \cdot \Phi \ = \ v^m \iota_{\partial_m} \Phi + \zeta_n \, \mathrm{d}x^n \wedge \Phi$$

w/ v: vector $\quad$ ζ: one-form

Define the annihilator of spinors as

$$L_\Phi \ \equiv \ \Big\{ v + \zeta \in F \oplus F^* \, \Big| \, (v + \zeta) \cdot \Phi = 0 \Big\}$$

$$\dim L_\Phi \ \leq \ 6$$

If $\dim L_\Phi = 6$ (maximally isotropic) $\rightarrow$ Φ is a pure spinor

Correspondence between pure spinors and generalized almost complex structures:

$$\mathcal{J} \leftrightarrow \Phi \qquad \text{if} \;\; L_{\mathcal{J}} = L_{\Phi} \quad \text{with} \;\; \dim L_{\Phi} = 6$$

More precisely: $\mathcal{J} \leftrightarrow$ a line bundle of pure spinor Φ

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Then, we can rewrite the generalized almost complex structure as

$$\boxed{\mathcal{J}_{\pm \Pi\Sigma} \; = \; \big\langle \mathrm{Re}\Phi_{\pm}, \Gamma_{\Pi\Sigma} \, \mathrm{Re}\Phi_{\pm} \big\rangle}$$

w/ Mukai pairing:

even forms: $\big\langle \Psi_+, \Phi_+ \big\rangle = \Psi_6 \wedge \Phi_0 - \Psi_4 \wedge \Phi_2 + \Psi_2 \wedge \Phi_4 - \Psi_0 \wedge \Phi_6$

odd forms: $\big\langle \Psi_-, \Phi_- \big\rangle = \Psi_5 \wedge \Phi_1 - \Psi_3 \wedge \Phi_3 + \Psi_1 \wedge \Phi_5$

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$\mathcal{J}$ is integrable $\longleftrightarrow$ $^{\exists}$ vector v and one-form ζ s.t. $\mathrm{d}\Phi = (v \llcorner + \zeta\wedge)\Phi$

generalized CY $\longleftrightarrow$ $^{\exists}\Phi$ is pure s.t. $\mathrm{d}\Phi = 0$

"twisted" GCY $\longleftrightarrow$ $^{\exists}\Phi$ is pure, and H is closed s.t. $(\mathrm{d} - H\wedge)\Phi = 0$

A spinor Φ can also be mapped to a bispinor by using

$$C \equiv \sum_k \frac{1}{k!} C^{(k)}_{m_1 \cdots m_k} \, \mathrm{d}x^{m_1} \wedge \cdots \wedge \mathrm{d}x^{m_k} \qquad \longleftrightarrow \qquad \not{C} \equiv \sum_k \frac{1}{k!} C^{(k)}_{m_1 \cdots m_k} \, \gamma_{\alpha\beta}^{m_1 \cdots m_k}$$

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On a geometry of a single $SU(3)$-structure, the following two $SU(3,3)$ spinors:

$$\Phi_{0+} = \eta_+ \otimes \eta_+^\dagger = \frac{1}{4} \sum_{k=0}^{6} \frac{1}{k!} \eta_+^\dagger \gamma_{m_1 \cdots m_k} \eta_+ \, \gamma^{m_1 \cdots m_k} = \frac{1}{8} \mathrm{e}^{-\mathrm{i}J}$$

$$\Phi_{0-} = \eta_+ \otimes \eta_-^\dagger = \frac{1}{4} \sum_{k=0}^{6} \frac{1}{k!} \eta_-^\dagger \gamma_{m_1 \cdots m_k} \eta_+ \, \gamma^{m_1 \cdots m_k} = -\frac{\mathrm{i}}{8} \Omega$$

Check purity: $(\delta + \mathrm{i}J)_m{}^n \gamma_n \eta_+ \otimes \eta_\pm^\dagger = 0 = \eta_+ \otimes \eta_\pm^\dagger \gamma_n (\delta \mp \mathrm{i}J)^n{}_m$

One-to-one correspondence: $\underline{\Phi_{0-} \leftrightarrow \mathcal{J}_1, \quad \Phi_{0+} \leftrightarrow \mathcal{J}_2}$

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One-to-one correspondence: $\underline{\Phi_{0-} \leftrightarrow \mathcal{J}_1, \quad \Phi_{0+} \leftrightarrow \mathcal{J}_2}$

On a generic geometry of a pair of $SU(3)$-structures defined by (η_+^1, η_+^2)

$$\begin{aligned} \Phi_{0+} &= \eta_+^1 \otimes \eta_+^{2\dagger} = \frac{1}{8} \big(\overline{c}_\parallel \mathrm{e}^{-\mathrm{i}j} - \mathrm{i} \overline{c}_\perp w \big) \wedge \mathrm{e}^{-\mathrm{i}v \wedge v'} \\ \Phi_{0-} &= \eta_+^1 \otimes \eta_-^{2\dagger} = -\frac{1}{8} \big(c_\perp \mathrm{e}^{-\mathrm{i}j} + \mathrm{i} c_\parallel w \big) \wedge (v + \mathrm{i}v') \end{aligned} \qquad |c_\parallel|^2 + |c_\perp|^2 = 1$$

$$\Phi_{\pm} \equiv \mathrm{e}^{-B}\Phi_{0\pm}$$

Each $\Phi_{\pm}$ defines an $SU(3,3)$ structure on E. Common structure is $SU(3) \times SU(3)$.

F is extended to E by including e^{-B}

Compatibility requires

$$\langle \Phi_+, V \cdot \Phi_- \rangle = \langle \overline{\Phi}_+, V \cdot \Phi_- \rangle = 0 \qquad \text{for } \forall V = x + \xi$$

$$\langle \Phi_+, \overline{\Phi}_+ \rangle = \langle \Phi_-, \overline{\Phi}_- \rangle$$

Start with a real form $\chi_f \in \wedge^{\text{even/odd}} F^*$ (associated with a real $Spin(6,6)$ spinor χ_s)

Regard χ_f as a stable form satisfying

$$\begin{aligned} q(\chi_f) &= -\frac{1}{4}\langle \chi_f, \Gamma_{\Pi\Sigma}\chi_f\rangle\langle \chi_f, \Gamma^{\Pi\Sigma}\chi_f\rangle \ \in \ \wedge^6 F^* \otimes \wedge^6 F^* \\ U &= \left\{\chi_f \in \wedge^{\text{even/odd}} F^* \,\middle|\, q(\chi_f) < 0\right\} \end{aligned}$$

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Define a Hitchin function

$$H(\chi_f) \equiv \sqrt{-\frac{1}{3} q(\chi_f)} \;\in\; \wedge^6 F^*$$

which gives an integrable complex structure on U

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$$

which gives an integrable complex structure on U

Then we can get another real form $\hat{\chi}_f$ and a complex form Φ_f by Mukai pairing

$$
\langle \hat{\chi}_f, \chi_f \rangle = -\mathrm{d}H(\chi_f) \qquad \text{i.e.,} \quad \hat{\chi}_f = -\frac{\partial H(\chi_f)}{\partial \chi_f}
$$

$$
\dashrightarrow \qquad \Phi_f \equiv \frac{1}{2}(\chi_f + \mathrm{i}\hat{\chi}_f) \qquad H(\Phi_f) = \mathrm{i}\langle \Phi_f, \overline{\Phi}_f \rangle
$$

Hitchin showed: Φ_f is a (form corresponding to) pure spinor!

N.J. Hitchin math/0010054, math/0107101, math/0209099

Consider the space of pure spinors Φ ...

$$\boxed{\begin{array}{ccc} \text{Mukai pairing } \langle *, * \rangle & \longrightarrow & \text{symplectic structure} \\ \text{Hitchin function } H(*) & \longrightarrow & \text{complex structure} \end{array}}$$

$$\downarrow$$

The space of pure spinor is Kähler

Consider the space of pure spinors Φ ...

Mukai pairing $\langle *, * \rangle$	$\longrightarrow$	symplectic structure
Hitchin function $H(*)$	$\longrightarrow$	complex structure

$\downarrow$

The space of pure spinor is Kähler

Compatible with $\Phi \to \lambda\Phi$ w/ $\lambda \in \mathbb{C}^*$

$\dashrightarrow$ The space becomes a local special Kähler geometry with Kähler potential K:

$$\exp(-K) \;=\; H(\Phi) \;=\; \mathrm{i}\big\langle \Phi, \overline{\Phi} \big\rangle \;=\; \mathrm{i}\big(\overline{X}^A \mathcal{F}_A - X^A \overline{\mathcal{F}}_A \big) \;\in\; \wedge^6 F^*$$

X^A : holomorphic projective coordinates

$\mathcal{F}_A$: derivative of prepotential $\mathcal{F}$ $\quad (\mathcal{F}_A = \partial \mathcal{F} / \partial X^A)$

Moduli spaces of $\mathcal{M}$ are special Kähler geometries of local type

Kähler potentials, prepotentials, projective coordinates

$$K_+ \;=\; -\log \mathrm{i} \int_{\mathcal{M}} \langle \Phi_+, \overline{\Phi}_+ \rangle \;=\; -\log \mathrm{i} \big(\overline{X}^A \mathcal{F}_A - X^A \overline{\mathcal{F}}_A \big)$$

$$K_- \;=\; -\log \mathrm{i} \int_{\mathcal{M}} \langle \Phi_-, \overline{\Phi}_- \rangle \;=\; -\log \mathrm{i} \big(\overline{Z}^I \mathcal{G}_I - Z^I \overline{\mathcal{G}}_I \big)$$

Expand the even/odd-forms $\Phi_\pm$ by the basis forms:

$$\Phi_+ \;=\; X^A \omega_A - \mathcal{F}_A \widetilde{\omega}^A \,, \qquad \omega_A = (1, \omega_a) \,, \qquad \widetilde{\omega}^A = (\widetilde{\omega}^a, \mathrm{vol}(\mathcal{M})) \qquad : \quad \text{0, 2, 4, 6-forms}$$

$$\Phi_- \;=\; Z^I \alpha_I - \mathcal{G}_I \beta^I \,, \qquad \alpha_I = (\alpha_0, \alpha_i) \,, \qquad \beta^I = (\beta^i, \beta^0) \qquad : \quad \text{1, 3, 5-forms}$$

$$\int_{\mathcal{M}} \langle \omega_A, \omega_B \rangle \;=\; 0 \,, \quad \int_{\mathcal{M}} \langle \omega_A, \widetilde{\omega}^B \rangle \;=\; \delta_A{}^B \,, \quad \int_{\mathcal{M}} \langle \alpha_I, \alpha_J \rangle \;=\; 0 \,, \quad \int_{\mathcal{M}} \langle \alpha_I, \beta^J \rangle \;=\; \delta_I{}^J$$

Contents

On generalized geometries with a single $SU(3)$-structure $(\eta^1_+ = \eta^2_+)$:

$$\begin{aligned} \mathrm{d}_H \omega_A &= {m_A}^I \alpha_I - e_{IA} \beta^I & \qquad \mathrm{d}_H \widetilde{\omega}^A &= 0 \\ \mathrm{d}_H \alpha_I &= e_{IA} \widetilde{\omega}^A & \qquad \mathrm{d}_H \beta^I &= {m_A}^I \widetilde{\omega}^A \end{aligned}$$

where NS three-form H deforms the differential operator:

$$\mathrm{d}H = 0 \,, \qquad H = H^{\mathrm{fl}} + \mathrm{d}B \,, \qquad H^{\mathrm{fl}} = {m_0}^I \alpha_I - e_{I0} \beta^I$$

$$\mathrm{d}_H \equiv \mathrm{d} - H^{\mathrm{fl}} \wedge$$

background	charges	
NS three-form flux	e_{I0}	${m_0}^I$
torsion	e_{Ia}	${m_a}^I$

On generalized geometries with $SU(3) \times SU(3)$ structures ($\eta_+^1 \neq \eta_+^2$ at some points):

Extend to the generalized differential operator $\mathcal{D}$:

$$\mathrm{d}_H \;=\; \mathrm{d} - H^{\mathsf{fl}}\wedge \quad \to \quad \mathcal{D} \;\equiv\; \mathrm{d} - H^{\mathsf{fl}} \wedge - f \cdot - Q \cdot - R \llcorner$$

$$\mathcal{D}\omega_A \;\sim\; m_A{}^I \, \alpha_I - e_{IA} \, \beta^I \qquad \mathcal{D}\widetilde{\omega}^A \;\sim\; -q^{IA} \, \alpha_I + p_I{}^A \, \beta^I$$

$$\mathcal{D}\alpha_I \;\sim\; p_I{}^A \, \omega_A + e_{IA} \, \widetilde{\omega}^A \qquad \mathcal{D}\beta^I \;\sim\; q^{IA} \, \omega_A + m_A{}^I \, \widetilde{\omega}^A$$

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$$\mathcal{D}\alpha_I \sim p_I{}^A \omega_A + e_{IA} \widetilde{\omega}^A \qquad \mathcal{D}\beta^I \sim q^{IA} \omega_A + m_A{}^I \widetilde{\omega}^A$$

The internal space becomes nongeometric:

$(f \cdot C)_{m_1 \cdots m_{k+1}}$	$\equiv$	$f^a{}_{[m_1 m_2} C_{\lvert a \rvert m_3 \cdots m_{k+1}]}$	(part of) structure const. in Gauged SUGRA
$(Q \cdot C)_{m_1 \cdots m_{k-1}}$	$\equiv$	$Q^{ab}{}_{[m_1} C_{\lvert ab \rvert m_2 \cdots m_{k-1}]}$	T-fold
$(R \llcorner C)_{m_1 \cdots m_{k-3}}$	$\equiv$	$R^{abc} C_{abc m_1 \cdots m_{k-3}}$	locally nongeometric background

Structure group = Diffeo. + duality trsf. $\dashrightarrow$ *Hull's Doubled formalism to study gauge symmetries*

backgrounds	flux charges			
Calabi-Yau	—			
Calabi-Yau with H	e_{I0}	${m_0}^I$		
generalized geometry w/ $SU(3)$	e_{IA}	${m_A}^I$		
generalized geometry w/ $SU(3) \times SU(3)$	e_{IA}	${m_A}^I$	$\boldsymbol{{p_I}^A}$	$\boldsymbol{q^{IA}}$

Contents

$$
\begin{aligned}
V &= \mathrm{e}^K \Big(K^{\mathcal{M}\overline{\mathcal{N}}} D_{\mathcal{M}} \mathcal{W} \, \overline{D_{\mathcal{N}} \mathcal{W}} - 3|\mathcal{W}|^2 \Big) + \frac{1}{2} |D^a|^2 \\
&\equiv V_{\mathcal{W}} + V_D
\end{aligned}
$$

Search of vacua $\partial_{\mathcal{P}} V \big|_* = 0$

$V_* > 0$: de Sitter space (non-SUSY)

$V_* = 0$: Minkowski space

$V_* < 0$: Anti-de Sitter space

$$\begin{aligned} V &= \mathrm{e}^K \Big(K^{\mathcal{M}\overline{\mathcal{N}}} D_{\mathcal{M}} \mathcal{W} \, \overline{D_{\mathcal{N}} \mathcal{W}} - 3 |\mathcal{W}|^2 \Big) + \frac{1}{2} |D^a|^2 \\ &\equiv V_{\mathcal{W}} + V_D \end{aligned}$$

Search of vacua $\partial_{\mathcal{P}} V \Big|_* = 0$

$V_* > 0$:	de Sitter space (non-SUSY)
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$$0 = \partial_{\mathcal{P}} V_{\mathcal{W}} = \mathrm{e}^K \Big\{ K^{\mathcal{M}\overline{\mathcal{N}}} D_{\mathcal{P}} D_{\mathcal{M}} \mathcal{W} \overline{D_{\mathcal{N}} \mathcal{W}} + \partial_{\mathcal{P}} K^{\mathcal{M}\overline{\mathcal{N}}} D_{\mathcal{M}} \mathcal{W} \overline{D_{\mathcal{N}} \mathcal{W}} - 2 \overline{\mathcal{W}} D_{\mathcal{P}} \mathcal{W} \Big\}$$

$$0 = \partial_{\mathcal{P}} V_D \quad \dashrightarrow \quad D^a = 0$$

Consider the SUSY condition $D_{\mathcal{P}} \mathcal{W} \equiv \big(\partial_{\mathcal{P}} + \partial_{\mathcal{P}} K \big) \mathcal{W} = 0$ in various cases.

Functionals are given by two Kähler potentials on two Hodge-Kähler geometries of $\Phi_\pm$:

$$\begin{aligned}
K &= K_+ + 4\varphi \\
K_+ &= -\log \mathrm{i} \int_{\mathcal{M}} \langle \Phi_+, \overline{\Phi}_+ \rangle \;=\; -\log \mathrm{i}\big(\overline{X}^A \mathcal{F}_A - X^A \overline{\mathcal{F}}_A\big) \\
K_- &= -\log \mathrm{i} \int_{\mathcal{M}} \langle \Phi_-, \overline{\Phi}_- \rangle \;=\; -\log \mathrm{i}\big(\overline{Z}^I \mathcal{G}_I - Z^I \overline{\mathcal{G}}_I\big) \\
\int_{\mathcal{M}} \mathrm{vol}_6 &= \frac{1}{8}\,\mathrm{e}^{-K_\pm} \;=\; \mathrm{e}^{-2\varphi + 2\phi^{(10)}}
\end{aligned}$$

Introduce $\mathcal{C} = \sqrt{2} ab\,\mathrm{e}^{-\phi^{(10)}} = 4ab\,\mathrm{e}^{\frac{K_-}{2} - \varphi}$

$$\begin{aligned}
\therefore \quad \mathrm{e}^{-2\varphi} &= \frac{|\mathcal{C}|^2}{16|a|^2|b|^2}\,\mathrm{e}^{-K_-} \;=\; \frac{\mathrm{i}}{16|a|^2|b|^2} \int_{\mathcal{M}} \langle \mathcal{C}\Phi_-, \overline{\mathcal{C}\Phi_-} \rangle \\
&= \frac{1}{8|a|^2|b|^2}\Big[\mathrm{Im}(\mathcal{C}Z^I)\mathrm{Re}(\mathcal{C}\mathcal{G}_I) - \mathrm{Re}(\mathcal{C}Z^I)\mathrm{Im}(\mathcal{C}\mathcal{G}_I)\Big]
\end{aligned}$$

SUSY variations yield the superpotential and the D-term:

$$\delta\psi_\mu = \nabla_\mu \varepsilon - \overline{n}^{\mathcal{A}} S_{\mathcal{A}\mathcal{B}} n^{*\mathcal{B}} \gamma_\mu \varepsilon^c \equiv \nabla_\mu \varepsilon - \mathrm{e}^{\frac{K}{2}} \mathcal{W} \gamma_\mu \varepsilon^c$$

$$\delta\chi^A = \mathrm{Im} F^A_{\mu\nu} \gamma^{\mu\nu} \varepsilon + \mathrm{i}\, D^A \varepsilon$$

$$\mathcal{W} = \frac{\mathrm{i}}{4\overline{a}b}\Big[4\mathrm{i}\, \mathrm{e}^{\frac{K_-}{2} - \varphi} \int_{\mathcal{M}} \big\langle \Phi_+, \mathcal{D}\mathrm{Im}(ab\Phi_-) \big\rangle + \frac{1}{\sqrt{2}} \int_{\mathcal{M}} \big\langle \Phi_+, G \big\rangle \Big]$$

$$\equiv \mathcal{W}^{\mathsf{RR}} + U^I \mathcal{W}_I^{\mathbb{Q}} + \widetilde{U}_I \widetilde{\mathcal{W}}_{\mathbb{Q}}^I$$

$$\mathcal{W}^{\mathsf{RR}} = -\frac{\mathrm{i}}{4\overline{a}b}\Big[X^A e_{\mathsf{RR}A} - \mathcal{F}_A m^A_{\mathsf{RR}} \Big]$$

$$\mathcal{W}_I^{\mathbb{Q}} = \frac{\mathrm{i}}{4\overline{a}b}\Big[X^A e_{IA} + \mathcal{F}_A p_I{}^A \Big], \qquad \widetilde{\mathcal{W}}_{\mathbb{Q}}^I = -\frac{\mathrm{i}}{4\overline{a}b}\Big[X^A m_A{}^I + \mathcal{F}_A q^{IA} \Big]$$

$$U^I = \xi^I + \mathrm{i}\,\mathrm{Im}(\mathcal{C} Z^I), \qquad \widetilde{U}_I = \widetilde{\xi}_I + \mathrm{i}\,\mathrm{Im}(\mathcal{C}\mathcal{G}_I)$$

$$D^A = 2\,\mathrm{e}^{K_+} (K_+)^{c\overline{d}} D_c X^A \overline{D_d X^B} \big[\overline{n}^{\mathcal{C}} (\sigma_x)_{\mathcal{C}}{}^{\mathcal{B}} n_{\mathcal{B}} \big] \Big(\mathcal{P}_B^x - \mathcal{N}_{BC} \widetilde{\mathcal{P}}^{xC} \Big)$$

1. *Set a simple prepotential:* $\mathcal{F} = D_{abc} \dfrac{X^a X^b X^c}{X^0}$
2. *Consider the simplest model:* single modulus t of Φ_+ (and U of Φ_-)

1. *Set a simple prepotential:* $\mathcal{F} = D_{abc} \dfrac{X^a X^b X^c}{X^0}$

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The superpotential is reduced to

$$\begin{aligned}
\mathcal{W} &= \mathcal{W}^{\mathsf{RR}} + U\, \mathcal{W}^{\mathbb{Q}} \\
\mathcal{W}^{\mathsf{RR}} &= m_{\mathsf{RR}}^0\, t^3 - 3\, m_{\mathsf{RR}}\, t^2 + e_{\mathsf{RR}}\, t + e_{\mathsf{RR}0} \\
\mathcal{W}^{\mathbb{Q}} &= {p_0}^0\, t^3 - 3\, p_0\, t^2 - e_0\, t - e_{00}
\end{aligned}$$

Consider the SUSY condition:

$$\begin{aligned}
D_t \mathcal{W} = 0 \dashrightarrow \quad 0 &= D_t \mathcal{W}^{\mathsf{RR}} + U\, D_t \mathcal{W}^{\mathbb{Q}} \\
D_U \mathcal{W} = 0 \dashrightarrow \quad 0 &= \frac{\mathrm{i}}{\mathrm{Im} U} \Big(\mathcal{W}^{\mathsf{RR}} + \mathrm{Re} U\, \mathcal{W}^{\mathbb{Q}} \Big)
\end{aligned}$$

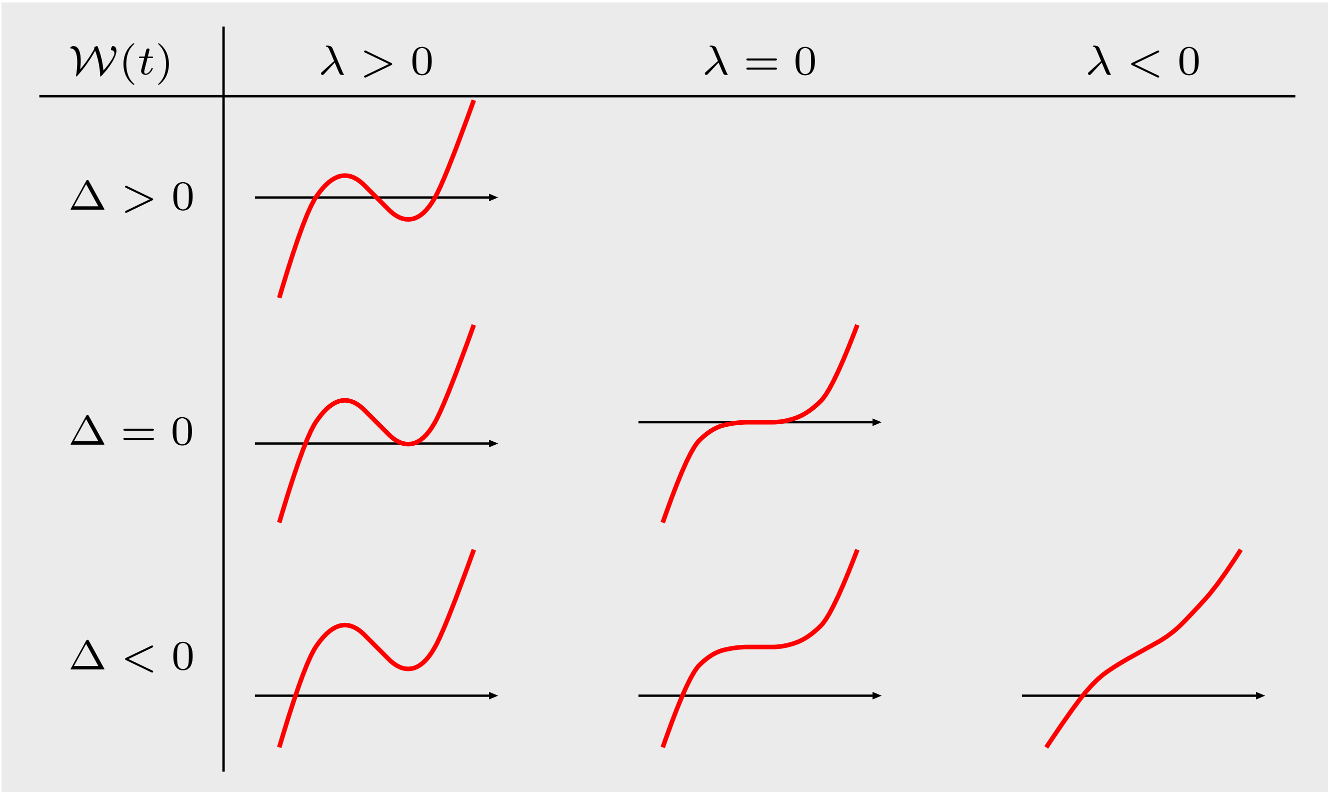
The discriminant of the superpotential $\mathcal{W}^{\mathsf{RR}}$ (and $\mathcal{W}^{\mathbb{Q}}$) governs the SUSY solutions.

▶ Discriminant of cubic equation

Consider a cubic function and its derivative: $\begin{cases} \mathcal{W}(t) &= a\,t^3 + b\,t^2 + c\,t + d \\ \partial_t \mathcal{W}(t) &= 3a\,t^2 + 2b\,t + c \end{cases}$

Discriminants $\Delta(\mathcal{W})$ and $\Delta(\partial_t \mathcal{W})$ are

$$\begin{aligned} \Delta(\mathcal{W}) \;&\equiv\; \Delta \;=\; -4b^3 d + b^2 c^2 - 4ac^3 + 18abcd - 27a^2 d^2 \\ \Delta(\partial_t \mathcal{W}) \;&\equiv\; \lambda \;=\; 4(b^2 - 3ac) \end{aligned}$$



$\Delta^{\mathsf{RR}} > 0$ case: always $\lambda^{\mathsf{RR}} > 0$, and exists a zero point: $D_t \mathcal{W}^{\mathsf{RR}} = 0$

$$
\begin{aligned}
D_t \mathcal{W}^{\mathsf{RR}}|_* &= 0 \\
t_*^{\mathsf{RR}} &= \frac{6\,(3\,m_{\mathsf{RR}}^0\,e_{\mathsf{RR}0} + m_{\mathsf{RR}}\,e_{\mathsf{RR}})}{\lambda^{\mathsf{RR}}} - 2\mathrm{i}\,\frac{\sqrt{3\,\Delta^{\mathsf{RR}}}}{\lambda^{\mathsf{RR}}} \\
\mathcal{W}_*^{\mathsf{RR}} &= -\frac{24\,\Delta^{\mathsf{RR}}}{(\lambda^{\mathsf{RR}})^3}\Big(36\,(m_{\mathsf{RR}})^3 + 36\,(m_{\mathsf{RR}}^0)^2 e_{\mathsf{RR}0} - 3\,m_{\mathsf{RR}}\lambda^{\mathsf{RR}} - 4\mathrm{i}\,m_{\mathsf{RR}}^0\sqrt{3\,\Delta^{\mathsf{RR}}}\Big)
\end{aligned}
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\end{aligned}
$$

$\Delta^{\text{RR}} < 0$ case: only $\lambda^{\text{RR}} < 0$ is physically allowed, and exists a zero point: $\mathcal{W}^{\text{RR}} = 0$

$$
\begin{aligned}
\mathcal{W}_*^{\text{RR}} &= m_{\text{RR}}^0(t_* - e)(t_* - \alpha)(t_* - \overline{\alpha}) \;=\; 0\,, \qquad t_* \;=\; \alpha^{\text{RR}} \;=\; \alpha_1 + \mathrm{i}\,\alpha_2 \\
\alpha_1 &= \frac{\lambda^{\text{RR}} + F^{2/3} + 12\,m_{\text{RR}}\,F^{1/3}}{12\,m_{\text{RR}}^0\,F^{1/3}} \\
(\alpha_2)^2 &= \frac{1}{m_{\text{RR}}^0}\Big(e_{\text{RR}} - 6\,m_{\text{RR}}\,\alpha_1 + 3\,m_{\text{RR}}^0\,(\alpha_1)^2\Big) \\
e &= -\frac{1}{m_{\text{RR}}^0}\Big(-3\,m_{\text{RR}} + 2\,m_{\text{RR}}^0\,\alpha_1\Big) \\
F &= 108\,(m_{\text{RR}}^0)^2 e_{\text{RR}0} + 12\,m_{\text{RR}}^0\sqrt{-3\Delta^{\text{RR}}} + 108\,(m_{\text{RR}})^3 - 9\,m_{\text{RR}}\,\lambda^{\text{RR}} \\
D_t \mathcal{W}^{\text{RR}}|_* &= 2\mathrm{i}\,m_{\text{RR}}^0(e - \alpha^{\text{RR}})\alpha_2
\end{aligned}
$$

... Analysis of $\mathcal{W}^{\mathbb{Q}}$ is also discussed.

Three types of solutions to satisfy $0 = D_t \mathcal{W}^{\mathsf{RR}} + U D_t \mathcal{W}^{\mathbb{Q}}$ and $0 = \mathcal{W}^{\mathsf{RR}} + \mathrm{Re} U \, \mathcal{W}^{\mathbb{Q}}$:

Three types of solutions to satisfy $0 = D_t \mathcal{W}^{\mathsf{RR}} + U D_t \mathcal{W}^{\mathbb{Q}}$ and $0 = \mathcal{W}^{\mathsf{RR}} + \mathrm{Re} U \, \mathcal{W}^{\mathbb{Q}}$:

- SUSY AdS vacuum: moduli are (almost) stabilized

$$\Delta^{\mathsf{RR}} > 0 \,, \quad \Delta^{\mathbb{Q}} > 0 \,; \quad D_t \mathcal{W}^{\mathsf{RR}}|_* = 0 = D_t \mathcal{W}^{\mathbb{Q}}|_*$$

$$t_*^{\mathsf{RR}} = t_*^{\mathbb{Q}} \,, \quad \mathrm{Re}\, U_* = -\frac{\mathcal{W}_*^{\mathsf{RR}}}{\mathcal{W}_*^{\mathbb{Q}}}$$

$$V_* = -3\, \mathrm{e}^K |\mathcal{W}_*|^2 = -\frac{4}{[\mathrm{Re}(\mathcal{C}\mathcal{G}_0)]^2} \sqrt{\frac{\Delta^{\mathbb{Q}}}{3}} \ll \mathcal{O}(1)$$

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- SUSY Minkowski vacuum: moduli are stabilized

$$\Delta^{\mathsf{RR}} < 0\,, \quad \Delta^{\mathbb{Q}} < 0\,; \quad \mathcal{W}_*^{\mathsf{RR}} = 0 = \mathcal{W}_*^{\mathbb{Q}}$$

$$\alpha^{\mathsf{RR}} = \alpha^{\mathbb{Q}}\,, \quad U_* = -\frac{D_t \mathcal{W}^{\mathsf{RR}}|_*}{D_t \mathcal{W}^{\mathbb{Q}}|_*} \neq 0$$

$$V_* = 0$$

Three types of solutions to satisfy $0 = D_t \mathcal{W}^{\mathsf{RR}} + U\, D_t \mathcal{W}^{\mathbb{Q}}$ and $0 = \mathcal{W}^{\mathsf{RR}} + \mathrm{Re} U\, \mathcal{W}^{\mathbb{Q}}$:

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$$V_* = -3\,\mathrm{e}^K |\mathcal{W}_*|^2 = -\frac{4}{[\mathrm{Re}(\mathcal{C}\mathcal{G}_0)]^2}\sqrt{\frac{\Delta^{\mathbb{Q}}}{3}} \ll \mathcal{O}(1)$$

- SUSY Minkowski vacuum: moduli are stabilized

$$\Delta^{\mathsf{RR}} < 0\,, \quad \Delta^{\mathbb{Q}} < 0\,; \quad \mathcal{W}_*^{\mathsf{RR}} = 0 = \mathcal{W}_*^{\mathbb{Q}}$$

$$\alpha^{\mathsf{RR}} = \alpha^{\mathbb{Q}}\,, \quad U_* = -\frac{D_t \mathcal{W}^{\mathsf{RR}}|_*}{D_t \mathcal{W}^{\mathbb{Q}}|_*} \neq 0$$

$$V_* = 0$$

- SUSY AdS vacua, but moduli t and U are not fixed: non-stabilized point

$$U = -\frac{D_t \mathcal{W}^{\mathsf{RR}}(t)}{D_t \mathcal{W}^{\mathbb{Q}}(t)}\,, \quad \mathrm{Re}\, U = -\frac{\mathcal{W}^{\mathsf{RR}}(t)}{\mathcal{W}^{\mathbb{Q}}(t)}$$

1. *Set* $e_{\mathsf{RR}A} = 0 = m_{\mathsf{RR}}^{A}$, $p_I{}^A = 0 = q^{IA}$, *and* single modulus t of Φ_+ (and U of Φ_-)
2. *Set a deformed prepotential:* $\mathcal{F} = \dfrac{(X^t)^3}{X^0}$

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Superpotential $\mathcal{W} = U\mathcal{W}^{\mathbb{Q}}$ with a simple setting $N_1 \neq 0$, $N_n = 0$:

$$\begin{aligned} D_t\mathcal{W}^{\mathbb{Q}} &= -e_{00} + \frac{3(t-\bar{t})^2 - \partial_t P}{(t-\bar{t})^3 - P}\Big(e_{00} + e_0\, t\Big) \\ P &\equiv -2\Big(N_1\, t^4 - \overline{N}_1\, \bar{t}^4 - 2N_1\, t^3\bar{t} + 2\overline{N}_1\, t\bar{t}^3\Big) \end{aligned}$$

SUSY condition

$D_t\mathcal{W} = D_U\mathcal{W} = 0$

has a solution

$$t^{\mathbb{Q}}_* = -\frac{2\,e_{00}}{e_0}, \qquad \mathrm{Re}\,U_* = 0$$

$$\mathcal{W}^{\mathbb{Q}}_* = e_{00}\,, \qquad \mathrm{Im}N_1 < 0$$

$$V_* = -3\,\mathrm{e}^K|\mathcal{W}_*|^2 = \frac{1}{[\mathrm{Re}(\mathcal{C}\mathcal{G}_0)]^2}\,\frac{3\,(e_0)^4}{16\,(e_{00})^2\,\mathrm{Im}N_1}$$

Also heterotic string on $SU(3)$-structure manifolds with torsion which carries α' corrections

Contents

Summary

- Studied generalized geometries and their applications to string compactifications
- Obtained a powerful rule to discuss SUSY vacua: Discriminants
- Exhibited that α' corrections are included in certain configurations

Discussions

- More generic configurations
- Gauge symmetries
- Understanding the physical interpretation of nongeometric fluxes

Thank You