Takesaki-Takai Duality for Crossed Products I: History and Contents

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History I

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- Takesaki succeeded to classify them completely by using the great new idea, so-called Takesaki Duality for W*-crossed products (72).
- At the almost same time, he conjectured that the duality also held for C*-crossed products.

History II

- Although his conjecture seemed to be similar to W^*-cases at a glance, it really contained a subtle problem comparing with Takesaki duality in W^*-crossed products . Actually, one had to establish a space free version by means of representations.

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- Fortunately, I then had some good informations from the paper of Zeller-Meier although the groups treated there were Discrete, which was insufficient solving his conjecture.
- I tried to generalize them to the case of Locally compact cases, and combining them with the preprint of Takesaki duality, I became to believe that his conjecture was correct.

History III

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- In a week or so, he found a big mistake in my paper. He explained me in detail that the way I found out was almost approaching to the ending, however the adjoint map I used there worked incorrectly reaching to the final stage. More precisely, the representation I found never splitted into a tensor product one by using the map cited above.

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- I checked repeatedly the part he mentioned, however I had never reached the conclusion for a long time.

History IV

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- I immediately followed his advice and executed finding both appropriate representation and unitary, and found them in a short time. Then I finally conquered the most difficult part of his problem.
- As soon as Prof.Takesaki checked the final version of my preprint, I sent it to both C.R.Acad.Paris and J.Func.Anal.,on which it was published (74,75).

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- After several months since I published it on C.R.Acad.Paris, I received the Ph.D thesis from M.Landstad who belonged to University of Pennsylvania, in which the same result of the duality for C*-crossed products was obtained.
- This means that If I had been struggling with the big mistake for half a year more, my present talk is probably never happened.

Contents I

- (Takesaki Conjecture): Let (A, G, α) be a C*-dynamical system with G a locally compact abelian group. Then there exists a dual C*-dynamical system $(A \rtimes_\alpha G, \hat{G}, \hat{\alpha})$ such that $(A \rtimes_\alpha G \rtimes_{\hat{\alpha}} \hat{G}, G, \widehat{\hat{\alpha}}) \simeq (A \otimes \mathbb{K}(\mathrm{L}^2(G)), G, \alpha \otimes \mathrm{Ad}(\lambda))$ as a C*-dynamical system, where $\mathbb{K}(\mathrm{L}^2(G))$ is the C*-algebra of all compact operators on $\mathrm{L}^2(G)$, and $\mathrm{Ad}(\lambda)$ is the adjoint action of the left translation λ of G on $\mathrm{L}^2(G)$.

Contents II

- Let us review briefly how to define C*-crossed product: let (A, G, α) be a C*-dynamical system where A is a C*-algebra, G is a locally compact unimodular group and $\alpha : G \to \mathrm{Aut}(A)$ is a homomorphism with the property that $||\alpha_g(a) - a|| \to 0 \; (g \to e)$ for all $a \in A$.

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- The C*-crosed product $A \rtimes_\alpha G$ is defined as the completion of the Banach *-algebra $\mathrm{L}^1(G, A)$ with: $xy(g) = \int_G x(h)\alpha_h(y(h^{-1}g))\, dh,\ x^*(g) = \alpha_g(x(g^{-1})^*)\ ,\ ||x||_1 = \int_G ||x(g)||\, dg$ with respect to the C*-norm $||x|| = \sup_{\pi \in \mathrm{L}^1(G,A)} ||\pi(x)||\ .$

Contents III

- Let G be amenable as a topological group, namely there exists a G-invariant mean on the C*-algebra $C^b(G)$ of all bounded continous complex valued functions on G with sup norm, or the dual space $\hat{G}$ of G is equal to its reduced one $\hat{G}_r$ as a topological space. For example, abelian, compact, nilpotent, and solvable groups are amenable, however the free groups $F_n, \ \mathrm{SL}(n, \mathbb{Z}) \ (n \geq 2)$ are non amenable.

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- Discrete amenability is much stronger than topological one: for instance, $SO(n)$ $(n \geq 3)$ are topologically amenable, but never discrete sense because $SO(n) \supset F_2$.

Contents IV

- Suppose G is amenable and A acts on a Hilbert space H by a representation π, then so does $A \rtimes_\alpha G$ on the Hilbert space $\mathrm{L}^2(G, H)$ by the induced representation $\mathrm{Ind}\,\pi$ of π, which is precisely defined by
$(\mathrm{Ind}\,\pi(x)\xi)(g) = \int_G \pi(\alpha_{g^{-1}}(x(h)))\xi(h^{-1}g)\,dh$
$(x \in \mathrm{L}^1(G, A), g \in G, \xi \in \mathrm{L}^2(G, H))$.

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$(x \in \mathrm{L}^1(G, A), g \in G, \xi \in \mathrm{L}^2(G, H))$.
- $\mathrm{Ind}\,\pi$ can be viewed as the integrated representation $\bar{\pi} \times \bar{\lambda}$ of the covariant one $(\bar{\pi}, \bar{\lambda})$ of (A, G) with the property that
$\bar{\lambda}_g \circ \bar{\pi}(a) \circ \lambda_{g^{-1}} = \bar{\pi} \circ \alpha_g(a)\ (a \in A,\ g \in G)$,where
$(\bar{\pi}(a))\xi(g) = \pi \circ \alpha_{g^{-1}}\xi(g),\ \bar{\lambda}_h\xi(g) = \xi(h^{-1}g)$

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  $(\bar{\pi}(a))\xi(g) = \pi \circ \alpha_{g^{-1}}\xi(g),\ \bar{\lambda}_h\xi(g) = \xi(h^{-1}g)$
- An important fact is that if π is faithful, so is $\mathrm{Ind}\,\pi$.

Contents V

- We now assume that G is a locally compact abelian group. Then we consider the the universal representation π of A, which is faithfully acting on the universal Hilbert space H. Then so is its induced one $\mathrm{Ind}\,\pi$ on the Hilbert space $\mathrm{L}^2(G, H)$.

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- We now assume that G is a locally compact abelian group. Then we consider the the universal representation π of A, which is faithfully acting on the universal Hilbert space H. Then so is its induced one $\mathrm{Ind}\,\pi$ on the Hilbert space $\mathrm{L}^2(G, H)$.
- Let us consider a new action $\hat{\alpha}$ of the Pontryagin dual group $\hat{G}$ of G on $A \rtimes_\alpha G$ defined by

$$\boxed{\hat{\alpha}_p(x)(g) = p(g)x(g)}$$

$(x \in \mathrm{L}^1(G, A), g \in G, p \in \hat{G})$. It serves a new C*-dynamical system $(A \rtimes_\alpha G,\ \hat{G},\ \hat{\alpha})$, which we call the dual C*-dynamical system of (A, G, α).

Contents VI

- We then consider its C*-crossed product, namely $A \rtimes_\alpha G \rtimes_{\hat{\alpha}} \hat{G}$. Since $\mathrm{Ind}\,\pi$ is faithful on $A \rtimes_\alpha G$, so is the second induced one $\mathrm{Ind}\,(\mathrm{Ind}\,\pi)$ acting on the Hilbert space $\mathrm{L}^2(\hat{G}, \mathrm{L}^2(G, H))$, which is denoted by π_1.

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- We next construct another representation of the C*-tensor product $A \otimes \mathbb{K}(\mathrm{L}^2(G))$ on the Hilbert space $H \otimes \mathrm{L}^2(G \times G)$ via the following process:

Contents VII

- let us take the translation action τ of G on the C*-algebra $C_0(G)$ of all complex valued continuous functions on G vanishing at infinity, which gives a C*-dynamical system $(C_0(G), G, \tau)$. We take its C*-crossed product $C_0(G) \rtimes_\tau G$.

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- Then we can check that

$$\operatorname{Ind} \delta_e(C_0(G) \rtimes_\tau G) = \mathbb{K}(\mathrm{L}^2(G))$$

where δ_e is the Dirac measure on $C_0(G)$ at the identity e of G.

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where δ_e is the Dirac measure on $C_0(G)$ at the identity e of G.
- Actually since $\oplus_{g\in G} \delta_e \circ \tau_g$ is faithful of $C_0(G)$ on $\bigoplus_{g\in G}\mathbb{C}$, so is $\mathrm{Ind}\, \delta_e$ of $C_0(G) \rtimes_\tau G$ on $\mathrm{L}^2(G)$.

Contents VIII

- Let us consider two C*-dynamical systems $(A \otimes C_0(G), G, \alpha \otimes \tau)$ and $(A \otimes C_0(G), G, \iota \otimes \tau)$. By taking the map:

$$\Phi(x)(g) = \alpha_g(x(g^{-1}h)),\ (\ x \in A \otimes C_0(G)\ )$$

these C*-dynamical systems are equivalent, so that

$$(A \otimes C_0(G)) \rtimes_{\alpha\otimes\tau} G \simeq (A \otimes C_0(G)) \rtimes_{\iota\otimes\tau} G$$

by the following isomorphism:

$$\tilde{\Phi}(x)(g) = \Phi(x(g)),\ (\ x \in \mathrm{L}^1(G, A \otimes C_0(G)\ )\ .$$

Contents IX

- We then consider another C*-dynamical system $(A \otimes C^*(\hat{G}), G, \alpha \otimes \chi)$, where χ is the character action of G on $C^*(\hat{G})$ defined by
$$\chi_g(f)(p) = p(g)f(g)$$
$(\ f \in \mathrm{L}^1(\hat{G}),\ g \in G,\ p \in \hat{G}\ )$.

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$(\ f \in \mathrm{L}^1(\hat{G}),\ g \in G,\ p \in \hat{G} \ )$.
- Then the two C*-dynamical systems $(A \otimes C^*(\hat{G}), G, \alpha \otimes \chi),\ (A \otimes C_0(G), G, \alpha \otimes \tau)$ are equivalent by the isomorphism $\mathcal{F}$ defined by
$$\mathcal{F} = \mathrm{Id}_A \otimes F \ ,$$
where F is the extended Fourier inverse isomorphism from $C^*(\hat{G})$ onto $C_0(G)$.

Contents X

- This implies that

$$(A \otimes C^*(\hat{G})) \rtimes_{\alpha\otimes\chi} G \simeq (A \otimes C_0(G)) \rtimes_{\alpha\otimes\tau} G$$

by the isomorphism $\tilde{\mathcal{F}}$ defined by

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- We then identify $(A \otimes C^*(\hat{G})) \rtimes_{\alpha \otimes \chi} G$ with $A \rtimes_\iota \hat{G} \rtimes_\beta G$,

where β is the action of G on $A \rtimes_\iota \hat{G}$ defined by

$$\beta_g(x)(p) = p(g)\alpha_g(x(p))$$

$(\; x \in \mathrm{L}^1(\hat{G}, A), \; g \in G, \; p \in \hat{G} \;).$

Contents XI

- We now study another double C*-crossed product $A \rtimes_\iota \hat{G} \rtimes_\beta G$ more precisely. As we have used before, let us take the double induced representation $\mathrm{Ind}(\mathrm{Ind}\,\pi)$ of $A \rtimes_\iota \hat{G} \rtimes_\beta G$ on the Hilbert space $\mathrm{L}^2(G, \mathrm{L}^2(\hat{G}))$, which we denote by π_2.

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- Fortunately, we can find a unitary U from $\mathrm{L}^2(\hat{G}, \mathrm{L}^2(G, H))$ onto $\mathrm{L}^2(G, \mathrm{L}^2(\hat{G}, H))$ such that $\pi_2(A \rtimes_\iota \hat{G} \rtimes_\beta G)$ is isomorphic to $\pi_1(A \rtimes_\alpha G \rtimes_{\hat{\alpha}} \hat{G})$ under $\mathrm{Ad}(U)$. Actually, the unitary U is gained by $U\xi(g,p) = \overline{p(g)}\xi(p,g)$, $(\ \xi \in \mathrm{L}^2(\hat{G} \times G, H),\ g \in G,\ p \in \hat{G}\ )$.

Contents XII

- Finally, we obtain that

$$\Pi \circ \widehat{\widehat{\alpha}} = (\alpha \otimes \mathrm{Ad}(\lambda)) \circ \Pi$$

where

$$\Pi = \{\pi_2 \circ \tilde{\mathcal{F}} \circ \tilde{\Phi} \circ (\mathrm{Id}_A \otimes \mathrm{Ind}\ \delta_e)\}^{-1} \circ \mathrm{Ad}(U) \circ \pi_1 \ .$$

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- Summing up the argument discussed above, we conclude that Takesaki Conjecture for C*-crossed products is affirmative.

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- Packer-Raeburn (89,90), Carey-Hannabuss-Mathai-McCann (98).

Contents XV

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- Bouwknegt-Evslin-Mahtai (03,04), Bouwknegt-Hannabuss-Mathai (04,05), Mathai-Rosenberg (05,06), Brodzki-Mathai-Rosenberg-Szabo (07,08), Szabo (08).

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Takesaki-Takai Duality for Crossed Products II: Applications

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Applications I

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- Let me remind all of you what this duality means once again:
  Let (A, G, α) be a C*-dynamical system with G a locally compact abelian group. Then there exists a dual C*-dynamical system $(A \rtimes_\alpha G, \hat{G}, \hat{\alpha})$ such that the double dual C*-dynamical system
  $(A \rtimes_\alpha G \rtimes_{\hat{\alpha}} \hat{G}, G, \widehat{\hat{\alpha}}) \simeq$ the tensor one
  $(A \otimes \mathbb{K}(\mathrm{L}^2(G)), G, \alpha \otimes \mathrm{Ad}(\lambda))$
  where $\mathbb{K}(\mathrm{L}^2(G))$ is the C*-algebra of all compact operators on $\mathrm{L}^2(G)$, and $\mathrm{Ad}(\lambda)$ is the adjoint action of the left translation λ of G on $\mathrm{L}^2(G)$.

Applications II

- One of the most important applications is perhaps **Thom isomorphism in K-theory** due to Connes (81), which precisely means that: let $(A, \mathbb{R}, \alpha)$ be a C*-flow, and denote by $\mathrm{K}_j(A)$ the K_j-group of A (j=0,1). Then he deduced using the duality for C*-crossed product that

$$\mathrm{K}_j(A \rtimes_\alpha \mathbb{R}) \simeq \mathrm{K}_{j+1}(A) \pmod{2}$$

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- Thom Isomorphism in Twisted K-theory also seems to be existed by using Quigg (resp. Bouwknegt-Hannabuss-Mathai)'s duality results for twisted (resp.induced) C^*-crossed products.

Applications III

- Using F^*-flows, the dual version of Connes'Thom isomorphism was proved by Elliott-Natsume-Nest. In other words, let $(A, \mathbb{R}, \alpha)$ be a F^*-flow. Then the following result holds:
  $\mathrm{H}_\lambda^{ev}(A \rtimes_\alpha \mathbb{R}) \simeq \mathrm{H}_\lambda^{od}(A)\ ,\ \mathrm{H}_\lambda^{od}(A \rtimes_\alpha \mathbb{R}) \simeq \mathrm{H}_\lambda^{ev}(A)$
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$\mathrm{H}_\lambda^{ev}(A \rtimes_\alpha \mathbb{R}) \simeq \mathrm{H}_\lambda^{od}(A)\ ,\ \mathrm{H}_\lambda^{od}(A \rtimes_\alpha \mathbb{R}) \simeq \mathrm{H}_\lambda^{ev}(A)$
where H_λ means the periodic cyclic cohomology (88).
- As their corollary, let G be a simply connected solvable Lie group (ex.$G = H_{\mathbb{R}}^{2n+1}, M_{\mathbb{R}}^{2n+1}$) and (A, G, α) be a $C^*(F^*)$-dynamical system. Then $\mathrm{K}_i(\mathrm{H}_\lambda^j)$-theory of $A \rtimes_\alpha G$ is isomorphic to $\mathrm{K}_{i+\dim G}(\mathrm{H}_\lambda^{j+\dim G})$-theory of A (mod 2), where $i = 0, 1,\ j = ev\ (od)$ respectively.

Applications IV

- We could state about it a little more than solvable cases. Namely, let $G = SO_0(n,1),\ SU(n,1)\ (n \geq 1)$ be the generalized Lorenz groups which are non amenable. But it is so-called K-amenable originally due to Cuntz (83) in discrete group cases, which was generalized to Lie group cases by Carey-Hannabuss-Mathai-McCann (98): Let G be a connected Lie group and K be its maximal compact subgroup such that G/K has a G-invariant spin^c structure, which induces the G-invariant Dirac operator ∂ on the Hibert space $H = \mathrm{L}^2(G/K, S)$ of all L^2-sections of the $\mathbb{Z}_2$-graded spinor bundles S over G/K.

Applications V

- Let $F = \partial(1+\partial^2)^{-1/2}$ be the pseudo-differential operator of order 0 acting on H, and M be the canonical representation of $C_0(G/K)$ on H defined by $M_f\xi = f\xi$. Then the triple (H, M, F) induces an element $\alpha_G \in \mathrm{KK}_G(C_0(G/K), \mathbb{C})$, which is called the Dirac element associated with G/K.

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- If G is semisimple, then there exists another canonical element $\beta_G \in \mathrm{KK}_G(\mathbb{C}, C_0(G/K))$ with the property that
  $\alpha_G \otimes_{\mathbb{C}} \beta_G = 1_{C_0(G/K)} \in \mathrm{KK}_G(C_0(G/K), C_0(G/K))$
  $\beta_G \otimes_{C_0(G/K)} \alpha_G = \gamma_G \in \mathrm{KK}_G(\mathbb{C}, \mathbb{C})$,
  which is called the Mishchenko element associated with G/K.

Applications VI

- Since G is semisimple, then the Killing form on G defines a G-invariant Riemannian metric of non-positive sectional curvature on G/K. Let $\mathcal{E} = C_0(G/K, S^*)$ be the G-invariant $C_0(G/K)$-module consisting of all continuous sections of the dual spinor bundles S^* vanishing at infinity, and F be a bounded operator on $\mathcal{E}$ defined as $F\xi(x) = c(v(x, x_0))\xi(x)$, where $v(x, x_0) \in T_x(G/K)$ is the unit vector which is tangent to the unique geodesic from x_0 to x and c means the Clifford multiplication. Then $v(x, x_0)$ is well defined outside a small neighborhood of x_0 can be extended continuously to all of G/K in any way.

Applications VII

- Then one sees that the triple $(\mathcal{E}, id, F)$ induces the element $\beta_G \in \mathrm{KK}(\mathbb{C}, C_0(G/K))$ with the property cited before.

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- (Kasparov 80) $SO_0(n, 1)$, and
  (Julg-Kasparov 95) $SU(n, 1)$ are K-amenable (n$\geq$ 1) .

Applications VIII

- (Carey-Hannabuss-Mathai-McCann 98) Let G be a K-amenable Lie group and K be its maximal compact subgroup. Suppose Γ is a lattice in G, then for any multiplier $\sigma \in \mathrm{H}^2(\Gamma, \mathrm{U}(1))$,

$$\mathrm{K}_*(C^*(\Gamma,\sigma)) \simeq \mathrm{K}^{*+\dim(G/K)}_{\delta(B_\sigma)}(\Gamma\backslash G/K)$$

where $C^*(\Gamma,\sigma)$ is the σ-twisted group C*-algebra of Γ, $\delta(B_\sigma) \in \mathrm{H}^3(\Gamma\backslash G/K)$ denotes the Dixmier-Douady invariant of a continuous trace C*-algebra B_σ, and $\mathrm{K}_{\delta(B_\sigma)}(\Gamma\backslash G/K)$ means the $\delta(B_\sigma)$-twisted K-theory of $\Gamma\backslash G/K$.

Applications VIV

- We finally apply it to the case of the fundamental groups Γ_g of compact Riemann surfaces Σ_g with genus $g \geq 2$. Let $G = SO_0(2,1),\ K = SO(2)$ and $\Gamma_g \subset G$. Then it follows that

Applications VIV

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- Given any $\sigma \in \mathrm{H}^2(\Gamma_g, \mathrm{U}(1))$, we have that
  $\mathrm{K}_0(C_r^*(\Gamma_g, \sigma)) \simeq \mathrm{K}^0(\Sigma_g) \simeq \mathbb{Z}^2$, and
  $\mathrm{K}_1(C_r^*(\Gamma_g, \sigma)) \simeq \mathrm{K}^1(\Sigma_g) \simeq \mathbb{Z}^{2g}$
  where $C_r^*(\Gamma_g, \sigma)$ means the reduced σ-twisted group C*-algebra of Γ_g
  (Carey-Hannabuss-Mathai-McCann 98, Natsume-Nest 99).

Applications X

- We next apply the duality for C*-crossed products (resp. twisted ones) to a new description of type II string **T-Duality** without (resp. with) NS H-flux. T-duality is actually a symmetry of string theory relating small and large distances, which is a generalization of the $R \to 1/R$ invariance of string theory compactified on a circle of radius R. In the case of superstring theory, type IIA-string theory can be shifted to type IIB-one and vice versa by T-duality. Moreover, type II and heterotic theories also mutually change under T-duality.

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- (Case I, Absence of NS H-flux): Let $X = M \times \mathbb{T}^n$ be a spacetime without a background H-flux.

Applications XI

- In this case, the T-dual space $\hat{X}$ is nothig but $\hat{X} = M \times \hat{\mathbb{T}}^n$, where $\hat{\mathbb{T}}^n = (\mathbb{R}^n)^*/\Lambda^*$ is the dual torus to $\mathbb{T}^n = \mathbb{R}^n/\Lambda$ where Λ is a lattice of maximal rank in $\mathbb{R}^n$ and Λ^* its dual lattice.

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- This situation is interpreted algebraically as follows: Let us take the $\mathbb{R}^n$-action φ on X by $\varphi|M = \mathrm{id}, \varphi|\mathbb{T}^n$=translations. Then $\hat{X} = \mathrm{Spec}(C_0(X) \rtimes_\varphi \mathbb{R}^n)$, because $C_0(X) \rtimes_\varphi \mathbb{R}^n$ is Morita equivalent to $C_0(\hat{X})$.

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- Using the duality for C*-crossed products, one knows that $\hat{\hat{X}} = X$, which implies that T-duality map:$X \to \hat{X}$ is invertible in $\mathrm{KK}(X, \hat{X})$.

Applications XII

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- Let $\mathcal{P}_0$ be the Poincaré line bundle over $\mathbb{T}^n \times \hat{\mathbb{T}}^n$, namely the unique line bundle with the property that $\mathcal{P}_0|_{\mathbb{T}^n \times \{\hat{t}\}} \in \mathrm{Pic}^0(\mathbb{T}^n)$ is the flat line bundle over $\mathbb{T}^n$ for all $\hat{t} \in \hat{\mathbb{T}}^n$ and $\mathcal{P}_0|_{\{0\} \times \hat{\mathbb{T}}^n} = \hat{\mathbb{T}}^n \times \mathbb{C}$.

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- Let $\mathcal{P} = p^*(\mathcal{P}_0)$ be the pull back bundle of $\mathcal{P}_0$ to $M \times \mathbb{T}^n \times \hat{\mathbb{T}}^n$, where p is the projection from $M \times \mathbb{T}^n \times \hat{\mathbb{T}}^n$ to $\mathbb{T}^n \times \hat{\mathbb{T}}^n$.

Applications XIII

- Then the T-duality isomorphsm $\mathrm{T}!$ from $\mathrm{K}^*(X)$ to $\mathrm{K}^{*+n}(\hat{X})$ is defined by
$$\mathrm{T}!(E) = p_2!(p_1^*(E) \otimes_{X\times\hat{\mathbb{T}}^n} \mathcal{P})$$
for all $E \in \mathrm{K}^*(X)$, where $p_1,\ p_2$ are the projection from $X \times \hat{\mathbb{T}}^n$ to $X, \hat{X}$ respectively, $p_1^*(E)$ is the pullback bundle of E to $X \times \hat{\mathbb{T}}^n$, and $p_2!$ is the pushforward map from $\mathrm{K}^*(X \times \hat{\mathbb{T}}^n)$ to $\mathrm{K}^*(\hat{X})$

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- Taking the transpose monomorphism p_1^* of p_1 from $C_0(X)$ into $C_0(X \times \hat{\mathbb{T}}^n)$, one easily has a KK-elements $p_1^* \in \mathrm{KK}^{*+n}(X, X \times \hat{\mathbb{T}}^n)$. Since p_2 is K-oriented, it follows from Karoubi-Kasparov that $p_2! \in \mathrm{KK}^*(X \times \hat{\mathbb{T}}^n, \hat{X})$.

Applications XIV

- Since the pullback bundle $\mathcal{P})$ of the Poincaré line bundle $\mathcal{P}_0$ to $X \times \hat{\mathbb{T}}^n$ can be considered as in $\mathrm{End}(\mathrm{K}^*(X \times \hat{\mathbb{T}}^n))$ by taking the Kasparov product, which is nothig more than $\mathrm{KK}^*(X \times \hat{\mathbb{T}}^n, X \times \hat{\mathbb{T}}^n)$ up to equivalence relations.

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- We then identify the T-duality map T! with

$$\mathrm{T}! = p_1^* \otimes_{X \times \hat{\mathbb{T}}^n} \otimes \mathcal{P} \otimes_{X \times \hat{\mathbb{T}}^n} p_2! \in \mathrm{KK}^{*+n}(X, \hat{X}) \ ,$$

(mod 2) , which is invertible with respect to the Kasparov product

Applications XV

- ((Case II, Presence of NS H-flux): Let us take $\mathbb{T} \to E \xrightarrow{\pi} M$ a principal $\mathbb{T}$-bundle over a smooth manifold M with a H-flux $[H] \in \mathrm{H}^3(E, \mathbb{Z})$. Then there exists a unique bundle $\mathcal{K} \to \mathcal{E} \to E$ whose Dixmier-Douady invariant $\delta(\mathcal{E}) = [H]$.So its continuous trace C*-algebra $\mathrm{CT}(\mathcal{E}, [H])$ satisfies $\delta(\mathrm{CT}(\mathcal{E}, [H])) = [H],\ \mathrm{Spec}(\mathrm{CT}(\mathcal{E}, [H])) = E$.

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- Let $\hat{E}$ be a T-dual of E, namely a principal $\hat{T}$-bundle over M : $\hat{T} \to \hat{E} \xrightarrow{\hat{\pi}} M$ such that its first Chern class $c_1(\hat{E}) = \pi!([H]),$ where $\pi!$ is the pushforward map: $\mathrm{H}^3(E, \mathbb{Z}) \to \mathrm{H}^2(M, \mathbb{Z})$.

Applications XVI

- Then one can defines a T-dual H-flux $[\hat{H}] \in \mathrm{H}^3(\hat{E}, \mathbb{Z})$ satisfying $c_1(E) = \hat{\pi}!(\hat{H})$ and $[H] = [\hat{H}] \in \mathrm{H}^3(E \times_M \hat{E}, \mathbb{Z})$, where $E \times_M \hat{E}$ is the fibered product of E and $\hat{E}$. Then it also induces the continuous trace C*-algebra $\mathrm{CT}(\hat{\mathcal{E}}, [\hat{H}])$ with the same property as stated before.

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- Since the fibers of E are $\mathbb{T}^1$, there exists a C*-dynamical system $(\mathrm{CT}(\mathcal{E}, [H]), \mathbb{R}, \varphi^*)$, where φ is the canonical action of $\mathbb{R}$ on E.

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- Since the fibers of E are $\mathbb{T}^1$, there exists a C*-dynamical system $(\mathrm{CT}(\mathcal{E}, [H]), \mathbb{R}, \varphi^*)$, where φ is the canonical action of $\mathbb{R}$ on E.
- Then its C*-crossed product $\mathrm{CT}(\mathcal{E}, [H]) \rtimes_{\varphi^*} \mathbb{R}$ is Morita equivarent to $\mathrm{CT}(\hat{\mathcal{E}}, [\hat{H}])$, which imples that $\hat{E} = \mathrm{Spec}(\mathrm{CT}(\mathcal{E}, [H]) \rtimes_{\varphi^*} \mathbb{R})$.

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- The T-dual mapping $\mathrm{T}! \ : \ E \to \hat{E}$ is also obtained as an invertible element of $\mathrm{KK}_1(\mathrm{CT}(\mathcal{E}, [H]), \mathrm{CT}(\hat{\mathcal{E}}, [\hat{H}]))$ by passing through a noncommutative Poincaré projective module over $\mathrm{CT}(E \times_M \hat{E})$.

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- What happens the T-duality phenomena in the case of $\mathbb{T}^n$-bundles for $n \geq 2$?

Applications XVI

- In these cases, quite difficult situation might be occurred. For instance, if $[H]|E_x = 0$ for all $x \in M$, then the $\mathbb{R}^n$-action φ on E can lift to $\mathcal{E}$, which induces an action φ^* on $\mathrm{CT}(\mathcal{E}, [H])$. Then the T-dual $\hat{E}$ of E can be defined by $\mathrm{Spec}(\mathrm{CT}(\mathcal{E}, [H]) \rtimes_{\varphi^*} \mathbb{R}^n)$, which is no longer of continuous trace classes. Actually, they are of continuous trace class if and only if $[x \in M \to [H]|E_x] = 0 \in \mathrm{H}^1(M, \mathrm{H}^2(\mathbb{T}^n, \mathbb{Z}))$.

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- In general cases, $\mathrm{CT}(\mathcal{E}, [H]) \rtimes_{\varphi^*} \mathbb{R}^n)$ may be viewed as the continuous sections of a continuous field of stable noncommutative n-tori on M. (Brodzki-Mathai-Rosenberg-Szabo 08).

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