素粒子

基本的な対称性 : Poincaré群 $\begin{cases} \text{Translation} \\ \text{Lorentz群} \begin{cases} \text{回転} \\ \text{boost} \end{cases} \end{cases}$

ある状態にPoincaré変換した状態は存在しなくてはならない

→ 状態は Poincaré群の表現 でなければならない

Poincaré群の既約表現 (physically relevant)

$\begin{cases} \text{massive} & P^2 > 0 \\ \text{massless} & P^2 = 0 \end{cases}$ $\qquad g_{\mu\nu} = \mathrm{diag.}(+1, -1, -1, -1)$

Massive $\quad P^2 = M^2 > 0$

$$W^2 = (\epsilon_{\mu\nu\rho\sigma} P^\nu M^{\rho\sigma})^2 \propto \underbrace{J}_{\text{spin}}{}^2 \quad (\text{in the rest frame})$$

Massiveな表現は mass と spin J で 決まる.

$(2J+1) \times \infty^3$ 個の状態

↑

$m = -J, -J+1, \cdots, J-1, J$

くりこみ可能性 → $J = 0, \frac{1}{2}, 1$

Massless $\quad P^2 = 0$

Spinは意味をもたない

helicity $(-J, J)$ $\quad$ (CPT) $\quad$ 1 for $J=0$

で表現が指定される $\qquad$ $2 \times \infty^3$ 個の状態.

(但し、慣習的に |helicity| を spin と呼ぶ)

Space-time symmetry → Lagrangian は scalar

Lagrangian の対称性

1. Spacetime symmetry (Lagrangian は Lorentz scalar)

   → momentum
   angular momentum } 保存

2. Internal symmetry

   2.1 Discrete sym

   2.2 Global sym (continuous) unitary 群の部分群

   2.3 Local (gauge) sym

3. Supersymmetry

スカラー場

$$p^2 - m^2 = 0$$

$$(-\partial^2 - m^2)\varphi = 0$$

$$\mathcal{L} = \frac{1}{2}\varphi\cancel{(-\partial^2 - m^2)}\varphi = \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{1}{2}m^2\varphi^2$$ (real scalar field)

$\dim\mathcal{L} = 4 \rightarrow \dim\varphi = 1$

Real vs. Complex scalar

a complex scalar = 2 real scalars

- Complex scalar is meaningful when there is a phase symmetry $\varphi \rightarrow e^{i\alpha}\varphi$

$$\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2) \qquad \varphi_i : \text{real}$$

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu\varphi_1)^2 - m^2\varphi_1^2] + \frac{1}{2}[(\partial_\mu\varphi_2)^2 - m^2\varphi_2^2]$$

$$= \partial_\mu\varphi^*\partial_\mu\varphi - m^2\varphi^*\varphi$$

Phase symmetry を課さなければ、任意の 2 real scalar 理論は complex scalar を使って書き直せる。

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu\varphi_1)^2 - m_1^2\varphi_1^2] + \frac{1}{2}[(\partial_\mu\varphi_2)^2 - m_2^2\varphi_2^2]$$

$$= \partial_\mu\varphi^*\partial_\mu\varphi - \frac{1}{2}(m_1^2 + m_2^2)\varphi^*\varphi - \frac{1}{4}(m_1^2 - m_2^2)(\varphi^2 + \varphi^{*2})$$

スカラー場理論の symmetry

n個の real scalar場 $\{\varphi_i\}$ $(i=1,\cdots,n)$ の理論を考える。

Maximal (global) symmetry = $O(n)$

∵) mass, interaction を無視すると

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{n}(\partial_\mu \varphi_i)^2$$

変換 $\varphi_i \to A_{ij}\varphi_j$ で この kinetic term を不変にするもの は $O(n) \sim SO(n) \times (\text{reflection})$ ■

Masses and interactions generally break the symmetry.

The most general renormalizable Lagrangian compatible with the $O(n)$ symmetry is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi_i)^2 - \frac{1}{2}m^2\varphi_i^2 - \frac{1}{4}\lambda(\varphi_i^2)^2$$

Complex field:

If there are $2n$ real fields, we can arrange them into n complex fields and restrict the symmetry to $U(n) \subset O(2n)$

表現の reality (1^{st} ver.)

表現 $[T^a, T^b] = i f^{abc} T^c$ 　　　　T^a : hermitian, f^{abc} : real

complex conjugate ↓ $[T^{a*}, T^{b*}] = -i f^{abc} T^{c*}$

↓ $[-T^{a*}, -T^{b*}] = +i f^{abc} (-T^{c*})$

$\{T^a\}$ が表現ならば $\{-T^{a*}\}$ も表現 (Conjugate rep.)

一般には異なる

The conjugate rep. is equivalent to the original rep.

⟺ if $\exists V$ (unitary) s.t. $V^\dagger T^a V = -T^{a*}$

この場合を real な表現 (広義の)

V が存在しない場合 complex な表現 と呼ぶ。

さらに、real 表現のうち、V の対称性によって

$$\begin{cases} V^T = V & \text{real} \\ V^T = -V & \text{pseudoreal} \end{cases}$$

real → このとき、$\{T^a\}$ を pure imaginary (表現行列を real) にできる

pseudoreal → できない

★ pseudoreal の例 　2 of $SU(2)$

$$[\tau^a, \tau^b] = 2i\epsilon^{abc}\tau^c$$

$$\tau^1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad \tau^2 = \begin{pmatrix} & -i \\ i & \end{pmatrix}, \quad \tau^3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$-\tau^{1*} = -\tau^1, \quad -\tau^{2*} = +\tau^2, \quad -\tau^{3*} = -\tau^3$$

$$V = \tau^2 \qquad V^\dagger(-\tau^{a*})V = \tau^a$$

$\{\tau^a\}$ は pure imaginary にとれない。

表現の reality (2^{nd} ver.)

任意の表現 n : $\varphi \to U\varphi$ $\qquad \varphi = \begin{pmatrix} \varphi^1 \\ \vdots \\ \varphi^n \end{pmatrix} \in \mathbb{C}^n$, $U \in U(n)$

Conjugate rep. n^* : $\varphi^* \to U^*\varphi^*$

or $\varphi^\dagger \to \varphi^\dagger U^\dagger$

1. 表現の積 $n \times n^*$ は必ず単位表現 1 と adjoint表現 ad を含む。

$$n \times n^* \supset 1 + \text{ad} + \cdots$$

$\because)$ $\varphi^\dagger\varphi \to \varphi^\dagger U^\dagger U\varphi = \varphi^\dagger\varphi$: 1

$\varphi^\dagger T^a \varphi$: ad $\quad$ (T^a: generator $\quad U = e^{i\theta^a T^a}$)

2. $n \times n$ は一般には 1 を含まない

$n \times n \not\supset 1$ $\qquad$ complex $\qquad n^* \not\sim n$

$(n \times n)_S \supset 1$ $\qquad$ real

$(n \times n)_A \supset 1$ $\qquad$ pseudoreal

$\Big\}\ n^* \sim n$

例.

3 of SU(3)	$3 \times 3 = 3^* + 6$ $3 \times 3^* = 1 + 8$	complex
3 of SO(3)	$3 \times 3 = 1_S + 3_A + 5_S$	real
2 of SU(2)	$2 \times 2 = 1_A + 3_S$	pseudoreal

Self-conjugate reps. (with detail)

表現 $R: \{T^a\} \xrightarrow{\text{Conjugate}} \{-T^{a*}\} : R^*$

Self-conjugate : $^\exists V$ (Unitary) $\quad V T^a V^\dagger = -T^{a*} \quad (*)$

$R \sim R^*$

このとき $V^T = \pm V \quad \begin{cases} + & \text{real} \\ - & \text{pseudoreal} \end{cases}$

証明

(*) の * $\quad V^* T^{a*} V^T = -T^a$

(*) と (*)* $\quad V^* V T^a V^\dagger V^T = T^a$

即 $\quad (V^* V) T^a = T^a (V^* V) \quad (^\forall a)$

$\{T^a\}$ 既約なら Schur's lemma により $\quad V^* V = \lambda \mathbb{1} \quad (**)$

即 $V = \lambda V^T$

Transpose $\quad V^T = \lambda V$

$V = \lambda V^T = \lambda^2 V$

$\therefore \lambda^2 = 1$

$V^T (**) V^\dagger \quad \mathbb{1} = \lambda V^T V^\dagger$

即 $\quad V^T V^\dagger = \lambda^{-1} \mathbb{1}$

$\mathrm{Tr}(**) \quad \mathrm{Tr}\, V^* V = \lambda \,\mathrm{Tr}\, \mathbb{1}$

$\mathrm{Tr}A = \mathrm{Tr}A^T \quad = \mathrm{Tr}\,(V^* V)^T = \mathrm{Tr}\, V^T V^\dagger = \lambda^{-1}\, \mathrm{Tr}\, \mathbb{1}$

即 $\quad \lambda = \lambda^{-1} \quad$ 即 $\quad \lambda^2 = 1 \quad$ 即 $\quad \lambda = \pm 1$

(QED)

(おまけ) $\quad \det V^* V = \det V^* \det V = |\det V|^2 = 1$

$= \det(\lambda \mathbb{1}) = \lambda^n \quad$ (nは表現の次数)

$\lambda^n = 1$. $\quad$ もしnが奇数なら $\lambda = 1$

奇数次元の pseudoreal 表現はない

$$R \text{ real} \leftrightarrow V^T = V \leftrightarrow (R\times R)_S \supset 1 \leftrightarrow (R\times R)_A \supset \text{adj}$$
$$R \text{ pseudoreal} \leftrightarrow V^T = -V \leftrightarrow (R\times R)_A \supset 1 \leftrightarrow (R\times R)_S \supset \text{adj}$$

(証明)

φ, φ' : 表現 R $\quad \delta\varphi = iT^a\varphi$ (same for φ')

transpose $\quad \delta\varphi^T = i\varphi^T T^{aT} = i\varphi^T T^{a*}$ (T^a は hermitian)

$\leftarrow V$ $\quad \delta\varphi^T V = i\varphi^T T^{a*} V$

$VT^aV^\dagger = -T^{a*}$ $\quad = -i(\varphi^T V)T^a$

$(\varphi^T V)$ は $\varphi^\dagger$ と同じ変換

Singlet

Remember $\delta(\varphi^\dagger\varphi') = 0$ $\quad$ ($R\times R^* \supset 1$)

$\Rightarrow \quad \delta(\varphi^T V \varphi') = 0$ (実際に計算してもよい)

$\varphi^T V \varphi' = \varphi_i \varphi'_j V_{ij}$ は singlet (invariant)

V が対称 $\rightarrow (R\times R)_S \supset 1$

V 反対称 $\rightarrow (R\times R)_A \supset 1$

Adjoint

Remember $\varphi^\dagger T^a \varphi$ は adjoint 表現

$\varphi^T V T^a \varphi' = \varphi_i \varphi'_j (VT^a)_{ij}$ は adjoint 表現

$V^T = \pm V$ なら $(VT^a)^T = T^{aT}V^T = \pm T^{a*}V = \pm(-VT^aV^\dagger)V = \mp VT^a$

逆符号

V が対称 $\rightarrow (R\times R)_A \supset \text{ad}$

反対称 $\rightarrow (R\times R)_S \supset \text{ad}$

例) SU(2) $\quad 2\times 2 = 1_A + \overset{\text{ad}}{3_S}$ $\quad$ (pseudoreal)

$\quad 3\times 3 = 1_S + 3_A + 5_S$ $\quad$ (real)

- Adjoint 表現は必ず real $\quad (T^a)_{bc} = -if^{abc}$ $\quad$ (f : real)

R real ↔ ∃ Group element が real になるような base が存在する
R pseudoreal ↔ 存在しない.

Group element が real ↔ generator $\{T^a\}$ が pure imaginary

V の対称性は表現の base によらない

$$T'^a = U T^a U^\dagger \quad (UU^\dagger = 1) \text{ のとき } \quad V' = U^* V U^\dagger$$

$$V = \pm V^T \quad \Leftrightarrow \quad V' = \pm V'^T$$

1. $\{T^a\}$ が pure imaginary なら V は対称 (R: real)

$$T^a = -T^{a*} \quad \rightarrow \quad V = 1 \text{ ととればよい.}$$

1. の対偶

pseudoreal 表現の $\{T^a\}$ が pure imaginary にとれるような base があったとすると $V(=1)$ は対称になり、もとの仮定 $(V = -V^T)$ と矛盾.
従って pseudoreal 表現の group element は real になりえない.

2. V が対称なら $\{T^a\}$ が pure imaginary にできる

V は unitary, symmetric → V は実直交行列で対角化できる (証明せよ!)

i.e. $O V O^T = D \quad OO^T = 1 \quad D = \mathrm{diag.}(e^{i\theta_1}, \cdots, e^{i\theta_n})$

$V = O^T D O$: $U = O^T D^{1/2} O$ とすると $\left(D^{1/2} = \mathrm{diag.}(e^{\frac{i}{2}\theta_1}, \cdots e^{\frac{i}{2}\theta_n}) \right)$

$U^2 = V$, $\quad U^\dagger U = 1$, $\quad U^T = U$ ← crucial!, $\quad (U^{-1} = U^\dagger = U^*)$

self-conjugate: $V T^a V^\dagger = -T^{a*} \rightarrow U^2 T^a U^{\dagger 2} = -T^{a*}$

$$\rightarrow \quad U T^a U^\dagger = -U^\dagger T^{a*} U$$
$$= -(U^T T^a U^*)^* = -(U T^a U^\dagger)^*$$

$\{U T^a U^\dagger\}$ は $\{T^a\}$ と同値で pure imaginary

スカラー場とsymmetry （注意）

Gauge/global symmetry があるとき、スカラー場は symmetry の（いくつかの）既約表現からなる

If 表現が real → real scalar 場が既約

complex にしても、real part と imaginary part は独立に変換するので 2 real と同等.

If 表現が pseudoreal / complex } → real scalar 場はとれない
Complex でなければならない

complex 表現の場合、ある表現 n のスカラー場が理論に存在することは、conjugate な表現 n^* のスカラー場があることと同等
（cf. fermion）

（例）

$$SO(10) \supset SU(5)$$

$$\underset{\text{real}}{10} = \underset{\text{complex}}{5 + 5^*}$$

scalar $\quad \phi^i \ (i=1,\cdots,10) \sim \varphi^\alpha \ (\alpha=1,\cdots,5)$ $\quad \varphi^\alpha : 5$, $\ \varphi^{*\alpha} : 5^*$

real 10成分 $\qquad$ complex 5成分

fermion $\quad \psi_L^i \quad i=1\cdots10 \sim \underset{5}{\psi_L^\alpha} + \underset{5^*}{\psi_L'^\alpha}$

フェルミオン (Spin-$\frac{1}{2}$)

$$\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

$$p^2-m^2 = (\not p+m)(\not p-m) \qquad \not p = p_\mu\gamma^\mu$$

$$(\not p-m)\psi = 0$$

$$\mathcal{L} = (\tfrac{1}{2})\,\bar\Psi(i\not\partial-m)\Psi$$

$\frac{1}{2}$ for Majorana

$$\dim\psi = \frac{3}{2} = \dim\varphi+\frac{1}{2}$$

$$\begin{tikzcd}
\text{Majorana} \arrow[rr, phantom, "\sim"] \arrow[dr, "\times 2"'] & & \text{Weyl} \arrow[dl, "\times 2"] \\
& \text{Dirac} &
\end{tikzcd}$$

Majorana: neutral, parity

Weyl: charged, parity ×

Dirac: charged, parity

$$\psi_{\text{Dirac}} = \psi_1 + i\psi_2$$

ψ_1, ψ_2 : Majorana $C\bar\Psi_i^T = \psi_i$

(or $\psi_i^* = \psi_i$ in Majorana rep.)

$$= \psi_L + \psi_R = \psi_L + C\bar\Psi_L'^T$$

Gamma 行列

$$\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2g^{\mu\nu}$$

$$\gamma_5\equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \qquad \gamma^\mu\gamma^5+\gamma^5\gamma^\mu=0$$

Dirac 表示

$$\gamma^0=\begin{pmatrix}1 & \\ & -1\end{pmatrix} \qquad \gamma^i=\begin{pmatrix} & \sigma_i \\ -\sigma_i & \end{pmatrix} \qquad \gamma_5=\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$

静止系 ($p=0$) で Dirac eq. 対角
Nonrelativistic limit で便利

Weyl 表示

$$\gamma^0=\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \qquad \gamma^i=\begin{pmatrix} & \sigma_i \\ -\sigma_i & \end{pmatrix} \qquad \gamma_5=\begin{pmatrix}-1 & \\ & 1\end{pmatrix}$$

Chiral projection $\frac{1-\gamma_5}{2}=\begin{pmatrix}1 & \\ & 0\end{pmatrix}$, $\frac{1+\gamma_5}{2}=\begin{pmatrix}0 & \\ & 1\end{pmatrix}$

$\left.\begin{array}{l}\text{High energy limit}\\ \text{Massless limit}\end{array}\right\}$ で便利

Majorana 表示

γ^μ が pure imaginary → Dirac 方程式が real
Majorana condition が trivial
(Majorana field が real に書ける)

Charge Conjugation

Scalar の場合

Real field (neutral boson)　　粒子 = 反粒子

π^0 : π^0を消す , π^0をつくる

$(\pi^0)^* = \pi^0$

$$\pi^0 \xrightarrow{\text{charge conjugation}} +\pi^0 \qquad (\text{cf. } K_1 \to -K_1)$$

Complex field (charged boson)　　粒子 ≠ 反粒子

π^+ : π^+を消す , π^-をつくる

$(\pi^+)^*$: π^+をつくる , π^-を消す

反粒子の場

π^- : π^-を消す , π^+をつくる

Charge conjugation

$$\pi^+ \xrightarrow{C} \pi^- = e^{i\eta}(\pi^+)^*$$

Charge conj. ~ complex conj.

フェルミオンと Charge Conjugation

electron field （粒子 = electron，反粒子 = positron）

e electronを消す．positronをつくる

$\bar{e}$ electronをつくる．positronを消す

positron field （粒子 = positron．反粒子 = electron）

e^c positronを消す．electronをつくる

$$(e^c)_\alpha = C_{\alpha\beta}\,\bar{e}_\beta$$

in matrix form

$$e^c = C\,\bar{e}^T$$

What is C ?

Requirement: positron field も 同じ形の Dirac eq. をみたす

$$(i\partial_\mu\gamma^\mu - m)\psi = 0 \qquad \text{Dirac eq.}$$

$$\psi^\dagger(-i\overleftarrow{\partial}_\mu\gamma^{\mu\dagger} - m) = 0 \qquad \text{h.c.}$$

$$\psi^\dagger\gamma^0(-i\overleftarrow{\partial}_\mu\gamma^0\gamma^{\mu\dagger}\gamma^0 - m) = 0 \qquad \leftarrow\gamma^0$$

$$\bar{\psi}(-i\overleftarrow{\partial}_\mu\gamma^\mu - m) = 0 \qquad \text{Dirac eq. for } \bar{\psi}$$

$$(-i\partial_\mu\gamma^{\mu T} - m)\bar{\psi}^T = 0 \qquad \text{transpose}$$

$$(-i\partial_\mu C\gamma^{\mu T}C^\dagger - m)\underbrace{C\bar{\psi}^T}_{\psi^c} = 0 \qquad C\to \quad (CC^\dagger = 1)$$

$$\boxed{C\gamma_\mu^T C^\dagger = -\gamma_\mu}$$

C = Candid

C の性質

(1) $C^\dagger C = 1$

(2) $C \gamma_\mu^T C^\dagger = -\gamma_\mu$

(3) $C^T = -C$ (4次元Minkowski) 証明省略

$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu}$

$\gamma^0 \gamma^{\mu\dagger} \gamma^0 = \gamma^\mu$

$\{\gamma_\mu\}$: Clifford algebra を generate :

base $\Gamma = \{ 1, \gamma_\mu, \sigma_{\mu\nu}, \gamma_\mu \gamma_5, \gamma_5 \}$

$\sigma_{\mu\nu} = i \gamma_\mu \gamma_\nu \quad (\mu \neq \nu)$

$= \frac{i}{2} [\gamma_\mu, \gamma_\nu]$

$\gamma_5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3$

(2) をくり返し使って一般に

(4) $$\boxed{C \Gamma^T C^\dagger = \varepsilon \Gamma}$$

$$\varepsilon = \begin{cases} +1 & \text{for } 1, \gamma_\mu\gamma_5, \gamma_5 \\ -1 & \text{for } \gamma_\mu, \sigma_{\mu\nu} \end{cases}$$

Charge conjugation

$\boxed{0}$ $\psi^c = C \bar{\psi}^T$

$\boxed{1}$ $(\psi^c)^c = \psi$

$\boxed{2}$ $\bar{\psi^c} = -\psi^T C^\dagger$

$\boxed{2}$ の証明

$\psi^c = C(\psi^\dagger \gamma^0)^T = C \gamma^{0T} \psi^*$

$\bar{\psi^c} \equiv \psi^{c\dagger} \gamma^0 = \psi^T \gamma^{0*} C^\dagger \gamma^0 = \psi^T \gamma^{0T} C^\dagger \gamma^0 = -\psi^T C^\dagger (\gamma^0)^2 = -\psi^T C^\dagger$

$(\gamma^{0\dagger} = \gamma^0 \rightarrow \gamma^{0*} = \gamma^{0T})$

フェルミオンの bilinear form と charge conjugation

$$\bar{\psi}\Gamma\psi = \varepsilon\,\bar{\psi}^c\Gamma\psi^c$$

$$\varepsilon = \begin{cases} +1 & S\ A\ P \\ -1 & V\ T \end{cases}$$

$$\because\quad \bar{\psi}\Gamma\psi = \bar{\psi}^{\alpha}\Gamma_{\alpha\beta}\psi^{\beta}$$

$$= -\psi^{\beta}\Gamma_{\alpha\beta}\bar{\psi}^{\alpha}$$ Fermi統計

$$= -\psi^{\beta}(\Gamma^T)_{\beta\alpha}\bar{\psi}^{\alpha}$$

$$= -\psi^T\Gamma^T\bar{\psi}^T$$

$$= -\psi^T C^{\dagger} C \Gamma^T C^{\dagger} C \bar{\psi}^T$$

$$= \bar{\psi}^c (C\Gamma^T C^{\dagger})\psi^c$$

$$= \varepsilon\,\bar{\psi}^c\Gamma\psi^c$$

意味.

$$\bar{\psi}\gamma_\mu\psi = -\bar{\psi}^c\gamma_\mu\psi^c$$ → 粒子と反粒子の charge は逆符号

$$\bar{\psi}\gamma_\mu\gamma_5\psi = +\bar{\psi}^c\gamma_\mu\gamma_5\psi^c$$ → axial charge は同符号

Majorana fermion の場合

$$\psi^c = \psi$$

$$(1-\varepsilon)\,\bar{\psi}\Gamma\psi = 0$$

$$\bar{\psi}\gamma_\mu\psi = 0$$ → Majorana粒子は neutral (chargeを持たない)

★ There is no C-odd Majorana onium states
(bound states of 2 identical Majorana fermion)

例. gluino−gluino bound states

J^{PC}	$S=0$	$S=1$
$L=0$	0^{-+}	$\cancel{1^{--}}$
$L=1$	$\cancel{1^{+-}}$	$0^{++}, 1^{++}, 2^{++}$
$L=2$	2^{-+}	$\cancel{1^{--}}, \cancel{2^{--}}, \cancel{3^{--}}$

Weyl fermion

Chiral projection

$$P_L = \frac{1-\gamma_5}{2} \qquad P_R = \frac{1+\gamma_5}{2}$$

$$P_L^2 = P_L \;, \quad P_R^2 = P_R \;, \quad P_L P_R = P_R P_L = 0$$

$$P_L + P_R = 1$$

2-component fermion

$$\psi = (P_L + P_R)\psi = P_L\psi + P_R\psi \equiv \psi_L + \psi_R$$

$$P_L\psi_L = \psi_L \,, \qquad P_R\psi_L = 0$$

$$P_R\psi_R = \psi_R \,, \qquad P_L\psi_R = 0$$

$$\bar{\psi}_L \equiv (\psi_L)^\dagger \gamma^0 = \psi^\dagger \frac{1-\gamma_5^\dagger}{2}\gamma^0 = \psi^\dagger\gamma^0 \frac{1+\gamma_5}{2} = \bar{\psi} P_R$$

$$\begin{aligned}(\psi_L)^c \equiv C\bar{\psi}_L^T = C(\bar{\psi}P_R)^T &= C\,\frac{1+\gamma_5^T}{2}\,\bar{\psi}^T \\ &= \frac{1+\gamma_5}{2}\,C\bar{\psi}^T \\ &= P_R\,\psi^c \\ &= (\psi^c)_R\end{aligned}$$

Weyl fermion & charge conjugation

Notation : $\psi^c = \psi\!\!\!\!\bot$. $e^c = \partial$ etc.

electron 2-component fields

e_L : left-handed electron を消す, right-handed positron をつくる
(helicity −)

e_R : e_R^- を消す. e_L^+ をつくる

positron

∂_L : e_L^+ を消す e_R^- をつくる

$e_R^\dagger \sim \partial_L$

In a theory without fermion number conservation, it is convenient to have all the fermion fields as left-handed, (As every GUT-man knows well)

because then we can treat all mass terms in a unified way. (Examples below)

For example, consider (e_L, ∂_L) instead of (e_L, e_R) as independent fields.

$$\partial_L = C\bar{e}_R^T$$

$$\bar{e}_R = -\partial_L^T C^\dagger$$

Dirac fermion

$$\mathcal{L} = \bar{\psi} i \not{\partial} \psi - m \bar{\psi}\psi$$

kinetic term

$$\begin{aligned}
\bar{\psi}\gamma_\mu\psi &= \bar{\psi}\gamma_\mu (P_L + P_R)\psi \\
&= \bar{\psi}\gamma_\mu P_L \psi + \bar{\psi}\gamma_\mu P_R \psi \\
&= \bar{\psi}\gamma_\mu \frac{1-\gamma_5}{2}\psi_L + \bar{\psi}\gamma_\mu \frac{1+\gamma_5}{2}\psi_R \\
&= \bar{\psi}\frac{1+\gamma_5}{2}\gamma_\mu\psi_L + \bar{\psi}\frac{1-\gamma_5}{2}\gamma_\mu\psi_R \\
&= \bar{\psi}_L\gamma_\mu\psi_L + \bar{\psi}_R\gamma_\mu\psi_R
\end{aligned}$$

$$\mathcal{L}_{kin} = \bar{\psi}_L i\not{\partial}\psi_L + \bar{\psi}_R i\not{\partial}\psi_R = \bar{\psi}_L i\not{\partial}\psi_L + \bar{\chi}_L i\not{\partial}\chi_L$$

mass term

$$\begin{aligned}
-\mathcal{L}_m &= m(\bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R) \\
&= m(-\chi_L^T C^\dagger \psi_L + \bar{\psi}_L C \bar{\chi}_L^T) \\
&= m(\chi_L \cdot \psi_L + h.c.)
\end{aligned}$$

Dirac = 2 left-handed Weyl

◇ Chirality structure of $\bar{\psi}\Gamma\psi$

$\Gamma = \gamma_\mu, \gamma_\mu\gamma_5$: chirality conserving (kinetic term, gauge coupling etc.

$\Gamma = 1, \gamma_5, \sigma_{\mu\nu}$: chirality changing (mass term, Yukawa coupling, magnetic moment etc.

Majorana fermion

Dirac (ψ_{1L}, ψ_{2L})

$$-\mathcal{L}_m = m(\psi_{1L}\cdot\psi_{2L} + h.c.)$$ Coupling of 2 Weyl fields

Majorana ψ_L

$$-\mathcal{L}_m = \frac{1}{2}m(\psi_L\cdot\psi_L + h.c.)$$ coupling of a Weyl field with itself

▷ Majorana fermion cannot have a conserved charge
(violated by the mass term)

→ If the neutrino is Majorana, lepton number is not conserved
(But the violation is proportional to the mass and can be very small)

▷ Majorana mass term $\psi_L\cdot\psi_L = -\psi_L^\alpha\psi_L^\beta C^\dagger_{\alpha\beta}$
is compatible with Fermi statistics
because C is antisymmetric
$$C_{\alpha\beta} = -C_{\beta\alpha}$$

▷ Only Weyl fermion in a real representation can have a invariant Majorana mass term

pseudoreal → Fermi statistics forbids Majorana mass

complex → no invariant mass term with itself

Weyl → Majorana

Chiral rep. $\gamma_5 = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$ $\quad P_L = \frac{1-\gamma_5}{2} = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}$

$P_R = \frac{1+\gamma_5}{2} = \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}$

$\psi_L = \frac{1-\gamma_5}{2}\psi_L = \begin{pmatrix} \xi \\ 0 \end{pmatrix}$ $\quad \xi$: 2成分

ξ^* は right-handed spinor. 但し $i\sigma_2\xi^*$ とした時 standard に変換する

$\psi \equiv \begin{pmatrix} \xi \\ i\sigma_2\xi^* \end{pmatrix}$ は Majorana spinor ($\psi^c = \psi$ をみたす)

もし ξ が charge をもっていると ξ^* は反対符号の charge を持つので
ψ は 0以外の 決まった charge を持てない

一般の表示では $\psi = \psi_L + C\bar{\psi}_L^T$ は Majorana spinor

Majorana × 2 = Dirac

Scalar field

mass term $-\frac{1}{2}m^2\varphi^2$

2 real scalars $-\frac{1}{2}m^2\varphi_1^2-\frac{1}{2}m^2\varphi_2^2=-\frac{1}{2}m^2(\varphi_1-i\varphi_2)(\varphi_1+i\varphi_2)=-m^2\varphi^*\varphi$
(with same mass)

$\varphi=\frac{1}{\sqrt{2}}(\varphi_1+i\varphi_2)$

2 real scalars (+ symmetry) = 1 complex scalar

fermions (2-component)

Majorana mass term $-\frac{1}{2}m\,\psi_L\cdot\psi_L$ + h.c. $\psi_L=\frac{1-\gamma_5}{2}\psi_L$

$\psi_L\cdot\psi_L\equiv-\psi_L^T C^{-1}\psi_L$ $C^T=-C$: antisymmetric ⟺ Fermi statistics.

2 Majorana fermions

$$-\frac{1}{2}m(\psi_{1L}\cdot\psi_{1L}+\psi_{2L}\cdot\psi_{2L}) + \text{h.c.}$$

$$\begin{cases}\psi_L=\frac{1}{\sqrt{2}}(\psi_{1L}+i\psi_{2L})\\ \psi_L'=\frac{1}{\sqrt{2}}(\psi_{1L}-i\psi_{2L})\end{cases}\qquad \begin{aligned}\psi_{1L}&=\frac{1}{\sqrt{2}}(\psi_L+\psi_L')\\ \psi_{2L}&=\frac{1}{\sqrt{2}i}(\psi_L-\psi_L')\end{aligned}$$

$$=-\frac{1}{2}m(\psi_L\cdot\psi_L'+\psi_L'\cdot\psi_L) + \text{h.c.}$$

$$=-m\,\psi_L'\cdot\psi_L + \text{h.c.}$$

$$\psi_R\equiv C\bar{\psi}_L'^{T}$$

$$=-m\,\bar{\psi}_R\psi_L + \text{h.c.}$$

$$=-m\bar{\psi}\psi$$

cf) Kinetic term

$$\bar{\psi}_{1L}i\not\partial\psi_{1L}+\bar{\psi}_{2L}i\not\partial\psi_{2L}$$
$$=\bar{\psi}_L i\not\partial\psi_L+\bar{\psi}_L' i\not\partial\psi_L'$$
$$=\bar{\psi}_L i\not\partial\psi_L+\bar{\psi}_R i\not\partial\psi_R$$
$$=\bar{\psi}i\not\partial\psi$$

2 Majorana fermions (+ symmetry) = 1 Dirac fermion

Massive Spin-1 粒子

$2J+1 = 3$ polarization states

静止系 → 一般の系

$$p = (M, 0, 0, 0) \xrightarrow{\text{boost}} (E, 0, 0, p) \qquad E = \gamma M,\quad p = \beta\gamma M$$

$$\epsilon_1 = (0, 1, 0, 0) \qquad (0, 1, 0, 0)$$
$$\epsilon_2 = (0, 0, 1, 0) \qquad (0, 0, 1, 0)$$
transverse

$$\epsilon_3 = (0, 0, 0, 1) \longrightarrow \left(\frac{p}{M}, 0, 0, \frac{E}{M}\right)$$
longitudinal

$$p \cdot \epsilon_i = 0$$
$$\epsilon_i \cdot \epsilon_j = -\delta_{ij}$$

At high energies, the longitudinal polarization vector grows as energy

Massive spin-1 → Massless spin-1

helicity

$$\begin{pmatrix} +1 \\ 0 \\ -1 \end{pmatrix} \quad \begin{array}{l} \longrightarrow \text{helicity } \pm 1 \quad (\text{massless vector}) \\ \longrightarrow \text{helicity } 0 \quad (\text{massless scalar}) \end{array}$$

Massive ベクトル場

$$Z^\mu \qquad Z_{\mu\nu} \equiv \partial_\mu Z_\nu - \partial_\nu Z_\mu$$

$\begin{cases} Z^0 : \text{scalar} \\ \mathbf{Z} : \text{vector} \end{cases}$

$$\mathcal{L} = -\frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} + \frac{1}{2} m^2 Z_\mu Z^\mu$$

eq. of motion $\begin{cases} (-\partial^2 - m^2) Z_\mu = 0 \\ \partial_\mu Z^\mu = 0 \end{cases}$

On mass shell ($p^2 = m^2$), eq. of motion kills the "scalar" part

3 independent polarization vectors

$$\epsilon_i^\mu \; (i = 1, 2, 3) \qquad p \cdot \epsilon = 0$$

$$\sum_{i=1}^{3} \epsilon_i^\mu \epsilon_i^{*\nu} = -g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}$$

Propagator

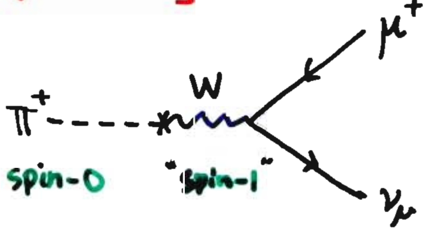
$$i \frac{-g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}}{p^2 - m^2 + i0}$$

▸ Off-shell ⇒ the scalar mode <u>does</u> propagate

if $p^2 > 0$, go to the "rest frame" $\quad p = (\sqrt{p^2}, 0, 0, 0)$

$$-g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2} = \begin{pmatrix} \frac{p^2 - m^2}{m^2} & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

(例) pion decay



くりこみ可能性

Interaction $\mathcal{L} = g\mathcal{O}$ (g: coupling , $\mathcal{O}$: field operator) を考える.

$\mathcal{O}$の次元を n とすると, gの次元は $4-n$

$\mathcal{O}$ が scalar , spinor 場 だけからなる場合

dim $\mathcal{O}$	dim g		
≤ 3	positive	superrenormalizable	(φ^3)
4	0	renormalizable	(φ^4, $\bar{\psi}\psi\varphi$)
≥ 5	negative	nonrenormalizable	($\bar{\psi}\psi\varphi^2$, $(\bar{\psi}\psi)^2$, ...)

(super)renormalizable な相互作用は 有限個しかない

$\because$) $\dim\varphi = 1$, $\dim\psi = \frac{3}{2}$ > 0

superrenormalizable : 発散する (sub) graph は 有限個

renormalizable : 発散する (sub) graph の type が 有限

(発散は 有限個の parameter に 吸収できる)

nonrenormalizable : 無限種類 の 発散

(no predictability)

厳密な意味での renormalizability : すべての発散が もとの Lagrangian の パラメータ (coupling, mass, wave fn.) の 再定義によって 吸収できる

scalar

4-point function (1PI, amputated)

dim  $= 0$

R $\lambda\varphi^4$ interaction

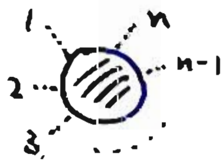
 $= \lambda$  $\sim \lambda^2 \int \frac{d^4k}{(k^2)^2} \sim \lambda^2 \ln\Lambda$

 $\lambda^m \int \frac{d^{4\ell}k}{k^{4\ell}} \sim \lambda^m \ln\Lambda$ (ℓ : # of loops)

SR $\mu\varphi^3$ $(\dim\mu = 1)$ $\to \mu^m \int \frac{d^{4\ell}k}{k^{4\ell+m}}$ = UV convergent

NR $\frac{1}{M}\varphi^5$ $\to \frac{1}{M^m} \int \frac{d^{4\ell}k}{k^{4\ell-m}} \sim \left(\frac{\Lambda}{M}\right)^m$ divergent

n-point function

dim  $= 4-n$

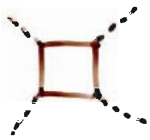
R $\lambda\varphi^4$ $\to \lambda^m \Lambda^{4-n}$ convergent for $n \geq 5$

SR $\mu\varphi^3$ $\to \mu^m \Lambda^{4-n-m}$ convergent for suff. large n or m.

NR $\frac{1}{M}\varphi^5$ $\to \frac{1}{M^m} \Lambda^{4-n+m}$ divergent for any n

くりこみ可能な理論ではすべての可能な発散が現れる

(例) Yukawa  $\rightarrow$ φ^4

φ^4  $\rightarrow$ φ^2

これらの発散をくりこむためには (bare) Lagrangian に対応する相互作用項が必要.

$\rightarrow$ massless φ^4 theory は unnatural

例外

1. Symmetry

(例) φ^4 $\rightarrow$ no φ^3 due to $\varphi \rightarrow -\varphi$ symmetry

2. Supersymmetry

 +  $= \Lambda^2 - \Lambda^2$

Spin-1 粒子の相互作用

Higher spin → More problem

massless vector

gauge symmetry must be respected

くりこみ可能な相互作用は (minimal) gauge interaction のみ

massive vector

Not all dim-4 interactions are renormalizable
because the propagator has a bad high-energy behavior.

くりこみ可能な理論

$\begin{cases} \text{spontaneously broken gauge theory} \\ \text{massive QED} \end{cases}$

Gauge interaction

1. Obtained by the replacement $\partial_\mu \to D_\mu$

2. Universality

   There is only one coupling constant (for each simple group)

   (However: $U(1)$)

   The gauge interaction of a particle is totally determined by knowing the representation of the particle.

3. Conserves fermion chirality

$$\bar{\psi}\gamma_\mu(v - a\gamma_5)\psi \cdot A_\mu$$
$$= (v+a)\,\bar{\psi}_L\gamma_\mu\psi_L + (v-a)\,\bar{\psi}_R\gamma_\mu\psi_R \cdot A_\mu$$

   Breaking of fermion chirality is entirely due to a mass term or a Yukawa interaction.

   (However, anomaly)

   must be absent for a gauge current

4. Nonrenormalizable effective interaction of gauge boson can be constructed from D_μ and $F_{\mu\nu}$

   ex.) Fermion anomalous moment $\quad \bar{\psi}\sigma_{\mu\nu}\psi F^{\mu\nu}$

   Fermion "seagull" $\quad \bar{\psi}\gamma_\mu D_\nu\psi F^{\mu\nu}$

Parity as a low-energy symmetry

理論全体がパリティを保存するかどうかにかかわらず、長距離力はパリティを保存する。

(仮定) 長距離力は unbroken gauge interaction

(no massless scalars / fermions

(証明)

gauge interaction

$$(g_L \bar{\psi}_L \gamma_\mu \psi_L + g_R \bar{\psi}_R \gamma_\mu \psi_R) A_\mu$$

fermion mass

$$m(\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L)$$

Fermion mass term が gauge invariant であるためには、ψ_L と ψ_R が同じ表現に属していなければならない。

→ Gauge interaction の universality により $g_L = g_R$ ∎

There is no surprise that parity was believed to be conserved before Lee-Yang

LIST OF POSSIBLE INTERACTIONS

spin	0	$\frac{1}{2}$	1
1	gauge $\lvert D_\mu\varphi\rvert^2$	gauge $\bar{\psi}\not{D}\psi$	gauge F^2
$\frac{1}{2}$	Yukawa $\bar{\psi}\psi\varphi$	NO	
0	scalar φ^3 , φ^4		

only one well tested experimentally

superrenormalizable

Prescription for model building

1. Fix the gauge group

gauge bosons determined

parameters : gauge couplings (# of simple & U(1) factors)

2. Fix the representations of fermions and scalars

gauge interactions of matter particle fixed

(no new parameters).

the total fermion rep. must be anomaly free

3. Give global symmetries if needed

4. Write down all possible mass terms and interactions compatible with the symmetries

(scalar potential and Yukawas)

parameters : scalar potential parameters ($\varphi^2, \varphi^3, \varphi^4$)
fermion masses
Yukawa couplings